A Robust A Posteriori Error Estimate for Elliptic Non-Homogeneous Equations

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Abstract: In this paper we present a new *a posteriori* error estimate for strongly heterogeneous elasticity problems. This new approach is based on a simple modification of the well known residual estimate, but with the nice property that it is correctly dimensionalised with respect to the physical data.

Key-words: *A posteriori* error estimate, Poisson's equation, linear elasticity, residuals, heterogeneity.

(Résumé : tsvp)

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Un Estimateur d'Erreur A Posteriori Robuste pour les Problèmes Elliptiques Non Homogènes

Résumé : Dans ce travail on présente un nouvel estimateur d'erreur *a posteriori* pour des problèmes d'élasticité avec coefficients élastiques fortement hétérogènes. La nouvelle approche, qui est une variation de l'estimateur par résidu, a la propriété d'être correctement dimensioné par rapport aux données physiques .

Mots-clé : Estimateur d'erreur *a posteriori*, équation de Poisson, problème elliptiques fortement hétérogènes, residus.

Contents

1 Introduction

Recent accidents have clearly demonstrated that reliable *a posteriori* error estimates and mesh adaption techniques were imperatively needed when computing large scale structures. From the theoretical point of view, this problem can apparently be solved either by using consistent residual estimates or by solving local auxiliary equilibrium problems at the element level. When used on real industrial problems, these theoretical strategies are faced with three main difficulties:

- the constitutive materials are usually anisotropic and heterogeneous,
- most engineering codes use second order or higher order elements, for which the theoretical tools are harder to implement and to derive,
- conforming mesh refinement is limited by many engineering and geometric constraints.

The purpose of this report is to describe and study a new version of local *a posteriori* error estimates. This estimate uses a weighted measure of element and interface residuals, and can be proved to be correctly dimensionalised with respect to the physical data, and to be uniformly valid with respect to material heterogeneities. For simplicity, the technique is introduced and analyzed for a simple Poisson type equation discretized by triangular or tetrahedric finite element grids. It is then tested numerically on more generic elasticity problems involving, for example, reinforced heterogeneous beams.

For more details on error estimates see [2] and the extensive bibliography cited therein. Other importants works are [4], [5], [11], [19],[22], [23], [24] and [25].

2 Model problem and notation

2.1 The continuous problem

Let Ω be a bounded domain of $\mathbb{R}^n$ (n= 2 or 3), with Lipschitz continuous boundary $\Gamma = \Gamma_D \cup \Gamma_N$, $\Gamma_D \cap \Gamma_N = \emptyset$. Let $f \in \mathrm{L}^2(\Omega)$ and $g \in \mathrm{H}^{-1/2}(\Gamma_D)$ be given data , and consider the following model problem

$$
(\mathrm{P}) \qquad \left\{ \begin{aligned} -\mathrm{div}(\kappa \nabla u) = &\ f \quad \text{in } \Omega, \\ u = &\ 0 \quad \text{on } \Gamma_D, \\ \kappa \frac{\partial u}{\partial n} = &\ g \quad \text{on } \Gamma_N. \end{aligned} \right.
$$

Here, the scalar coefficient κ is supposed to be piecewise constant. In the simplest case, this means that the domain Ω is split into two subdomains Ω_1 and Ω_2 with interface Γ_{12} (see Fig 1) and that we have

$$
\kappa := \left\{ \begin{array}{ll} \kappa_1 & \text{on } \Omega_1, \\ \kappa_2 & \text{on } \Omega_2, \end{array} \right.
$$

with $\kappa_1, \kappa_2 > 0$.

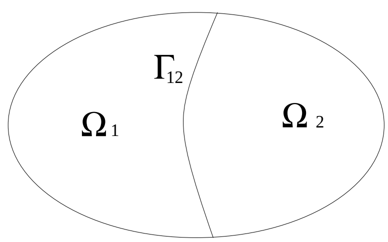


Figure 1: $\Omega = \Omega_1 \cup \Gamma_{12} \cup \Omega_2$.

The standard weak formulation of the problem (P) (see [10]) is then:
Find $u \in \mathrm{H}$ *such that*

$$
a(u, v) = < F, v > \quad , \forall v \in \mathrm{H}, \tag{2.1}
$$

where

$$
\mathrm{H} \ := \ \{ v \in \mathrm{H}^1(\Omega) \, / \, v = 0 \text{ on } \Gamma_D \},
$$

$$a(u,v) := \int_{\Omega_1} \kappa_1 \nabla u \cdot \nabla v + \int_{\Omega_2} \kappa_2 \nabla u \cdot \nabla v,$$
$$< F, v > := \int_{\Omega} f v + \int_{\Gamma_N} g v\,.$$

This space is endowed with the natural *energy norm*

$$||v||_{e,\Omega} := \sqrt{a(v,v)} = \{\sum_{T\in\mathcal{T}_h} (\sqrt{\kappa_T}|v|_{1,2,T})^2\}^{1/2} \quad , \forall v \in \mathrm{H}\,, \tag{2.2}$$

with $|v|_{1,2,T}$ denoting the usual semi-norm

$$|v|^2_{1,2,T} = \int_T |\nabla v|^2 d\Omega.$$

2.2 Finite element discretization

Let h be a positive discretization parameter, and consider a triangulation $\mathcal{T}_h$ of $\bar{\Omega}$, that is a partition of $\bar{\Omega}$ into non degenerate triangles T (resp. tetrahedra in dimension n=3), with diameter bounded by h, and such that each pair of elements T_1 and T_2 of $\mathcal{T}_h$ are either disjoint or share a vertex, an edge or a complete face. We denote by h_T the diameter of T, by ρ_T the diameter of the circle (resp. sphere) inscribed in T and we set

$$\sigma_T := \frac{h_T}{\rho_T}\,.$$

We assume that the family of triangulations $(\mathcal{T}_h)_h$ is *shape regular*, i.e. there exists a constant σ, independent of h, such that

$$\sigma_T \le \sigma \qquad , \forall T \in \mathcal{T}_h\,. \tag{2.3}$$

On each element T we then introduce a local finite element space $\mathbb{P}_k(T)$ of polynomial functions defined on the element T and with degree less than or equal to k. With this notation, we define the finite element space H_h by

$$\mathrm{H}_h := \{v_h \in \mathcal{C}(\Omega)\,/\, v_h = 0 \text{ on } \Gamma_D\,, \forall T \in \mathcal{T}_h\ , v_h|_T \in \mathbb{P}_k(T)\,\} \quad (k \ge 1)\,.$$

Then the approximate problem of (2.1) is:

Find $u_h \in \mathrm{H}_h$ such that

$$a(u_h, v_h) =< F, v_h > \quad , \forall v_h \in \mathrm{H}_h\,. \tag{2.4}$$

2.3 Edges and vertices

For any $T \in \mathcal{T}_h$ we denote by $\mathcal{E}(T)$ and $\mathcal{N}(T)$ the set of its faces and vertices, respectively (see [20]), and set

$$\mathcal{E}_{h,\Omega} := \bigcup_{T \in \mathcal{T}_h} \mathcal{E}(T) \quad , \quad \mathcal{N}_{h,\Omega} := \bigcup_{T \in \mathcal{T}_h} \mathcal{N}(T) .$$

We split $\mathcal{E}_{h,\Omega}$ and $\mathcal{N}_{h,\Omega}$ into

$$\mathcal{E}_{h,\Omega} = (\mathcal{E}_h \backslash \mathcal{E}_{12}) \cup \mathcal{E}_{12} \cup \mathcal{E}_N \cup \mathcal{E}_D \quad , \quad \mathcal{N}_{h,\Omega} = \mathcal{N}_h \cup \mathcal{N}_N \cup \mathcal{N}_D ,$$

with

$$\begin{aligned} \mathcal{E}_N &:= \{E \in \mathcal{E}_{h,\Omega} \,/\, E \subset \Gamma_N\} \quad , \quad \mathcal{E}_D := \{E \in \mathcal{E}_{h,\Omega} \,/\, E \subset \Gamma_D\} , \\ \mathcal{E}_{12} &:= \{E \in \mathcal{E}_{h,\Omega} \,/\, E \subset \Gamma_{12}\} , \\ \mathcal{N}_N &:= \{x \in \mathcal{N}_{h,\Omega} \,/\, x \in \Gamma_N\} \quad , \quad \mathcal{N}_D := \{x \in \mathcal{N}_{h,\Omega} \,/\, x \in \Gamma_D\} . \end{aligned}$$

Given an $E \in \mathcal{E}_{h,\Omega}$ we denote by $\mathcal{N}(E)$ the set of its vertices. For $T \in \mathcal{T}_h$, $E \in \mathcal{E}_{h,\Omega}$ and $x \in \mathcal{N}_{h,\Omega}$ we define their neighborhoods (see Fig. 2)

$$\begin{aligned} w_T &:= \bigcup_{\mathcal{E}(T) \cap \mathcal{E}(T') \neq \emptyset} T' \, , \; w_E := \bigcup_{E \in \mathcal{E}(T')} T' \, , \; w_x := \bigcup_{x \in \mathcal{N}(T')} T' , \\ \tilde{w}_T &:= \bigcup_{\mathcal{N}(T) \cap \mathcal{N}(T') \neq \emptyset} T' \, , \; \tilde{w}_E := \bigcup_{\mathcal{N}(E) \cap \mathcal{N}(T') \neq \emptyset} T' , \end{aligned}$$

and for $E \in \mathcal{E}_{12}$ we define the interface neighborhood

$$s_E := \{E' \in \mathcal{E}_{h,\Omega} \,/\, E' \subset \Gamma_{12} \text{ and } \mathcal{N}(E) \cap \mathcal{N}(E') \neq \emptyset\} .$$

Remark Condition (2.3) implies that the ratios h_T/h_E, $T \in \mathcal{T}_h$, $E \in \mathcal{E}(T)$, and $h_T/h_{T'}$, $T, T' \in \mathcal{T}_h$, $\mathcal{N}(T) \cap \mathcal{N}(T') \neq \emptyset$, are bounded from below and from above by constants which only depend on σ.

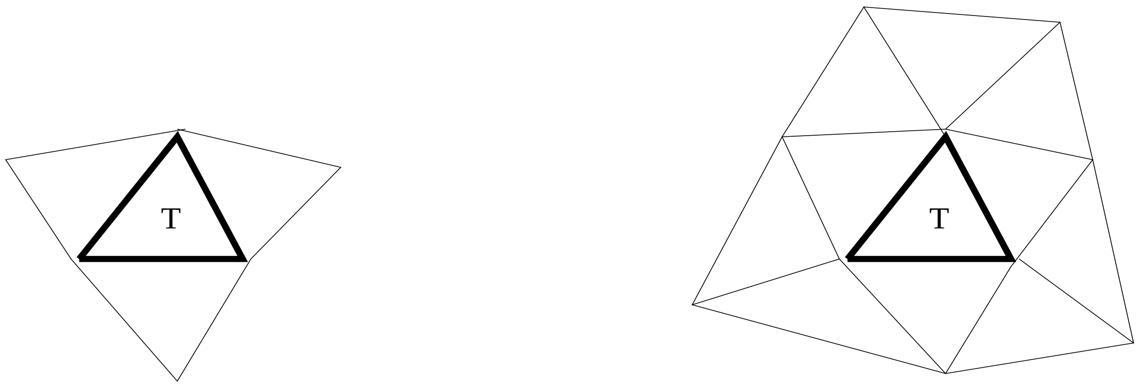


Figure 2: w_T and $\tilde{w}_T$, respectively.

2.4 Bubble functions

For each element $T \in \mathcal{T}_h$ we can define the element *bubble* function b_T by

$$b_T := \begin{cases} 27\lambda_{T,1}\lambda_{T,2}\lambda_{T,3} & \text{on } T\,, \\ 0 & \text{on } \Omega\backslash T\,, \end{cases}$$

in two dimensions, and by

$$b_T := \begin{cases} 256\lambda_{T,1}\lambda_{T,2}\lambda_{T,3}\lambda_{T,4} & \text{on } T, \\ 0 & \text{on } \Omega\backslash T\,, \end{cases}$$

in three dimensions. Above $\lambda_{T,j}(M)$ denote the j *barycentric coordinates* of the point M in T. Similarly, to each face $E \in \mathcal{E}_{h,\Omega}$, we can define the face bubble function

$$b_E := \begin{cases} 4\lambda_{T_i,1}\lambda_{T_i,2} & \text{on } T_i, i = 1, 2 \\ 0 & \text{on } \Omega\backslash w_E\,, \end{cases}$$

if $E \in \mathcal{E}_h$ and $w_E = T_1 \cup T_2$, or

$$b_E := \begin{cases} 4\lambda_{T,1}\lambda_{T,2} & \text{on } T \\ 0 & \text{on } \Omega\backslash w_E\,, \end{cases}$$

if $E \in \mathcal{E}_N \cup \mathcal{E}_D$ and $w_E = T$. The above definition of b_E assumes that we are in two space dimensions, and that in each triangle T_i, the edge E is associated to the vertices with local numbers 1 and 2. In three dimensions, the construction of b_E is the same within the replacement of the quadratic product $4\lambda_{T_i,1}\lambda_{T_i,2}$ by the cubic product $27\lambda_{T_i,1}\lambda_{T_i,2}\lambda_{T_i,3}$.

By construction, we have the followings properties of the bubble functions b_T and b_E

Lemma 1 *Let $T \in \mathcal{T}_h$ and $E \in \mathcal{E}_{h,\Omega}$ be arbitrary, then*

$$\operatorname{supp} b_T \subset T, \quad 0 \leq b_T \leq 1, \quad \max_{x \in T} b_T(x) = 1, \tag{2.5}$$

$$\operatorname{supp} b_E \subset w_E, \quad 0 \leq b_E \leq 1, \quad \max_{x \in E} b_E(x) = 1\,. \tag{2.6}$$

Moreover, using standard discrete norm equivalence arguments, we can prove

Lemma 2 *For any local function $f \in \mathbb{P}_{k-2}(T)$ or $g \in \mathbb{P}_{k-1}(E)$, there exist positive constants C_1, C_2, C_3, C_4, C_5, with C_1 and C_2 only depending on the polynomial degree k and C_3, C_4 or C_5 depending both of k and of the shape regularity σ, such that*

$$\begin{aligned}
C_1 \|f\|^2_{0,2,T} &\leq \int_T b_T f^2 \leq C_2 \|f\|^2_{0,2,T}, \\
C_1 \|g\|^2_{0,2,E} &\leq \int_E b_E g^2 \leq C_2 \|g\|^2_{0,2,E}, \\
\int_T |\nabla(b_T f)|^2 &\leq C_3 h_T^{-2} \int_T f^2, \\
\int_{T \in w_E} |\nabla(b_E g)|^2 &\leq C_4 h_E^{-1} \int_E g^2, \\
\int_{T \in w_E} |b_E g|^2 &\leq C_5 h_E \int_E g^2.
\end{aligned}$$

Proof All proofs make a linear change of variable $x = B\hat{x} + b$ from the present element T or face E on the unit reference element $\hat{T}$ or face $\hat{E}$.

With such change of variable, we first have with obvious notation

$$\int_T b_T f^2 dx = \int_{\hat{T}} \hat{b}_T \hat{f}^2 |\frac{dx}{d\hat{x}}| d\hat{x} \quad , \forall \hat{f} \in \mathbb{P}_{k-2}(\hat{T}),$$

$$\int_T f^2 dx \;=\; \int_{\hat T} \hat f^2 |\frac{dx}{d\hat x}| d\hat x \quad ,\forall \hat f \in \mathbb{P}_{k-2}(\hat T).$$

But, by construction, $\hat b_T$ is strictly positive inside $\hat T$. Hence, $\int_{\hat T} \hat b_T \hat f^2 d\hat x$ defines a norm on $\mathbb{P}_{k-2}(\hat T)$, and thus is equivalent to any other norm on $\mathbb{P}_{k-2}(\hat T)$ and in particular to the L^2-norm. We therefore have

$$C_2 \int_{\hat T} \hat f^2 d\hat x \geq \int_{\hat T} \hat b_T \hat f^2 d\hat x \geq C_1 \int_{\hat T} \hat f^2 d\hat x,$$

which once multiplied by the constant value $|\frac{dx}{d\hat x}| = |B|$ yields the desired equivalence. The proof is absolutely similar for $\int_E b_E g^2$. Observe that since $|b_T| \leq 1$ and $|b_E| \leq 1$, we have that both C_1 and C_2 are less or equal to 1. For $k = 1$ on triangles, we have for example $C_1 = C_2 = \frac{9}{20}$.

Similarly, a direct change of variables yields

$$\begin{aligned}\int_T |\nabla(b_T f)|^2 dx \;&=\; \int_{\hat T} |\hat\nabla(\hat b_T \hat f)\cdot B^{-1}|^2 |\frac{dx}{d\hat x}| d\hat x \\ &\leq\; \sigma^2 h_T^{-2} |\frac{dx}{d\hat x}| \int_{\hat T} |\hat\nabla(\hat b_T \hat f)|^2 d\hat x.\end{aligned}$$

Again, the last term defines a norm on $\mathbb{P}_{k-2}(\hat T)$ and hence

$$|\frac{dx}{d\hat x}| \int_{\hat T} |\hat\nabla(\hat b_T \hat f)|^2 d\hat x \leq C_k |\frac{dx}{d\hat x}| \int_{\hat T} \hat f^2 d\hat x \leq C_k \int_T f^2 dx.$$

We also have in the same way

$$\begin{aligned}\int_T |\nabla(b_E g)|^2 dx \;&\leq\; \sigma^2 h_T^{-2} |\frac{dx}{d\hat x}| \int_{\hat T} |\hat\nabla(\hat b_E \hat g)|^2 d\hat x \\ &\leq\; C_k' \sigma^2 h_T^{-2} |\frac{dx}{d\hat x}| \int_{\hat E} \hat g^2 d\hat l \quad ,\forall \hat g \in \mathbb{P}_{k-1}(\hat E) \\ &\leq\; C_k' \sigma^2 h_T^{-2} |\frac{dx}{d\hat x}||\frac{d\hat l}{dl}| \int_E g^2 dl \\ &\leq\; C_k' \sigma^2 h_T^{-2} h_E^3 \sigma^2 h_E^{-2} \int_E g^2 dl \\ &\leq\; C_4 h_E^{-1} \int_E g^2 dl \quad ,\forall g \in \mathbb{P}_{k-1}(E),\end{aligned}$$

and the rest of the lemma follows ■

Lemma 3 *There exists a projection operator* $R_h : \mathrm{H} \rightarrow \mathrm{H}_h$ *such that for all* $v \in \mathrm{H}$

$$\begin{aligned} ||v - R_h v||_{0,2,T} &\leq C_6 h_T |v|_{1,2,\tilde{w}_T \cap \Omega_i} \\ ||v - R_h v||_{0,2,E} &\leq C_7 h_E^{1/2} |v|_{1,2,\tilde{w}_E \cap \Omega_i} \ , \forall i = 1, 2 \end{aligned} \tag{2.7}$$

where $T \in \mathcal{T}_h$, $E \in \mathcal{E}_N \cup \mathcal{E}_h$ *and the positive constants* C_6, C_7 *are independent on* h.

Proof Such result is a nontrivial extension of a local projection result proved in [8]. It requires that the local distance $||v - R_h v||$ be controlled only by the local norm of $|v|$ not on the full patch $\tilde{w}_T$, but on its restriction $\tilde{w}_T \cap \Omega_i$ on the subdomain Ω_i. The proof [17], to be outlined in a forthcoming paper, uses a local average of v_h around each node in order to define the values of $R_h v$ at each node ■

3 A posteriori error estimates

3.1 Construction of the estimate

The purpose of this section is to propose a local explicit evaluation of the error between the exact solution u of our original problem (P) and the approximate solution u_h of the finite element problem (2.4). This error estimate should be easy to compute, should only involve the data and the approximate solution u_h, and its efficiency should be independent of the choice of the physical parameters κ_i. As classically observed in the literature (see [1], [4], [6] and [7]), the dual energy norm of the residual gives a good indication of the error. The challenge is then to obtain an explicit local approximation of this norm, uniformly valid with respect to the coefficients κ_i.

For this purpose, on each face $E \in \mathcal{E}_h$ separating the elements T_1 and T_2, we first introduce weighting factors $\alpha(T_i, E), i = 1, 2$, which satisfy

$$\alpha(T_1, E) + \alpha(T_2, E) = 1, \tag{3.1}$$

$$\frac{\alpha(T_1, E)^2}{\kappa_{T_1}} + \frac{\alpha(T_2, E)^2}{\kappa_{T_2}} \leq \frac{1}{\kappa_{T_i}}, \forall i = 1, 2, \tag{3.2}$$

where κ_{T_i} denotes the restriction of the physical coefficient κ to the element T_i. A good choice of coefficients is to take

$$\alpha(T_i, E) = \frac{\kappa_{T_i}}{\kappa_{T_1} + \kappa_{T_2}},$$

which obviously satisfies

$$\frac{\alpha(T_1, E)^2}{\kappa_{T_1}} + \frac{\alpha(T_2, E)^2}{\kappa_{T_2}} = \frac{1}{\kappa_{T_1} + \kappa_{T_2}}.$$

Next, we introduce the piecewise projections f_T and g_E of right hand sides f and g on each element or face subspace $\mathbb{P}_{k-2}(T), T \in \mathcal{T}_h$ or $\mathbb{P}_{k-1}(E), E \in \mathcal{E}_{h,\Omega}$ defined by

$$\int_T f_T q = \int_T f q \quad , \forall q \in \mathbb{P}_{k-2}(T), f_T \in \mathbb{P}_{k-2}(T), \tag{3.3}$$

$$\int_E g_E q = \int_E g q \quad , \forall q \in \mathbb{P}_{k-1}(E), g_E \in \mathbb{P}_{k-1}(E), \tag{3.4}$$

and the *weighted element residuals* $\eta_{R,T}$ given by

$$\begin{aligned}\eta_{R,T} \;&:=\; \Big\{\frac{h_T^2}{\kappa_T}||f_T+\kappa_T\Delta u_h||^2_{0,2,T}+\sum_{E\in\mathcal{E}(T)\cap\mathcal{E}_N}\frac{h_E}{\kappa_T}||g_E-\kappa_T\partial_n u_h||^2_{0,2,E}\\ &+\sum_{E\in\mathcal{E}(T)\cap\mathcal{E}_h}\frac{\alpha(T,E)^2}{\kappa_T}h_E||[\kappa_T\partial_n u_h]||^2_{0,2,E}\Big\}^{1/2}.\end{aligned} \tag{3.5}$$

Observe that the value of $\eta_{R,T}$ scales exactly like the solution energy norm when changing the physical scales or units. Above, we have used the standard notation for jumps in normal derivatives

$$[\kappa_T\partial_n u_h]=\kappa_{T_1}\partial_n u_h-\kappa_{T_2}\partial_n u_h.$$

With this notation we can prove

Theorem 4 *There exist two positive constants $\bar{C}$ and $\underline{C}$, independent of h and κ, such that*

$$\begin{aligned}||u-u_h||_{e,\Omega} \;&\leq\; \bar{C}\Big\{\sum_{T\in\mathcal{T}_h}\eta^2_{R,T}+\sum_{T\in\mathcal{T}_h}\frac{h_T^2}{\kappa_T}||f_T-f||^2_{0,2,T}\\ &+\sum_{E\in\mathcal{E}_N}\frac{h_E}{\kappa_T}||g_E-g||^2_{0,2,E}\Big\}^{1/2}\end{aligned} \tag{3.6}$$

and

$$\eta_{R,T}\leq\underline{C}\Big\{||u-u_h||^2_{e,w_T}+\sum_{T'\subset w_T}\frac{h_{T'}^2}{\kappa_{T'}}||f_T-f||^2_{0,2,T'}+\sum_{E\in\mathcal{E}(T)\cap\mathcal{E}_N}\frac{h_E}{\kappa_T}||g_E-g||^2_{0,2,E}\Big\}^{1/2}. \tag{3.7}$$

Proof

As usual, the proof is split into three parts : an algebraic manipulation of the residual, the derivation of the upper bound (3.6) , and of the lower bound (3.7).

Step 1 : residual transform.

By construction of the continuous and discrete problems (2.1) and (2.4), and after integration by parts on each element T, we can write for any v in H

$$\begin{aligned} a(u-u_h,v) &= \int_\Omega fv + \int_{\Gamma_N} gv - a(u_h,v) \\ &= \int_\Omega fv + \int_{\Gamma_N} gv - (\int_{\Omega_1} \kappa_1 \nabla u_h \cdot \nabla v + \int_{\Omega_2} \kappa_2 \nabla u_h \cdot \nabla v) \\ &= \sum_{T\in\mathcal{T}_h} \int_T (f+\kappa_T \Delta u_h) v + \sum_{E\in\mathcal{E}_N} \int_E (g-\kappa\partial_n u_h) v \\ &\quad - \sum_{E\in\mathcal{E}_h} \int_E [\kappa\partial_n u_h] v\,. \end{aligned} \tag{3.8}$$

Then using the fact that $a(u-u_h,v_h)=0\,,\forall v_h\in \mathrm{H}_h$ and the partition of unity (3.1) , we obtain

$$\begin{aligned} a(u-u_h,v) &= \sum_{T\in\mathcal{T}_h} \int_T (f+\kappa\Delta u_h)(v-v_h) + \sum_{E\in\mathcal{E}_N} \int_E (g-\kappa\partial_n u_h)(v-v_h) \\ &\quad - \sum_{E\in\mathcal{E}_h} \int_E [\kappa\partial_n u_h](v-v_h) \\ &= \sum_{T\in\mathcal{T}_h} \Big\{ \int_T \frac{1}{\sqrt{\kappa_T}} (f+\kappa_T\Delta u_h)\sqrt{\kappa_T}(v-v_h) \\ &\quad + \sum_{E\in\mathcal{E}(T)\cap\mathcal{E}_N} \int_E \frac{1}{\sqrt{\kappa_T}} (g-\kappa_T\partial_n u_h)\sqrt{\kappa_T}(v-v_h) \\ &\quad - \sum_{E\in\mathcal{E}(T)\cap\mathcal{E}_h} \int_E \frac{\alpha(T,E)}{\sqrt{\kappa_T}} [\kappa\partial_n u_h]\sqrt{\kappa_T}(v-v_h) \Big\}. \end{aligned} \tag{3.9}$$

Step 2: upper bound.

Let us now take $v_h = R_h v$. Using (2.2), (2.7) and the Cauchy-Schwarz inequality, the residual (3.9) can then be bounded by

$$a(u-u_h,v) \le \sum_i \sum_{T\in\mathcal{T}_h\cap\Omega_i} \Big(C_6 \frac{h_T}{\sqrt{\kappa_i}} ||f+\kappa_i\Delta u_h||_{0,2,T} \sqrt{\kappa_i} |v|_{1,2,\tilde{\omega}_T\cap\Omega_i}$$

$$
\begin{aligned}
&+C_7 \sum_{E\in\mathcal{E}_N\cap\mathcal{E}(T)} \frac{h_E^{1/2}}{\sqrt{\kappa_i}}||g-\kappa_i\partial_n u_h||_{0,2,E}\sqrt{\kappa_i}|v|_{1,2,\tilde{w}_E\cap\Omega_i}\\
&+C_7 \sum_{E\in(\mathcal{E}_h\backslash\mathcal{E}_{12})\cap\mathcal{E}(T)} \frac{\alpha(T,E)h_E^{1/2}}{\sqrt{\kappa_i}}||[\kappa_i\partial_n u_h]||_{0,2,E}\sqrt{\kappa_i}|v|_{1,2,\tilde{w}_E\cap\Omega_i}\Big)\\
&+C_7 \sum_{E\in\mathcal{E}_{12}}\sum_{E\subset T_i\subset\Omega_i} \frac{\alpha(T_i,E)h_E^{1/2}}{\sqrt{\kappa}_i}||\kappa_1\partial_n u_h-\kappa_2\partial_n u_h||_{0,2,E}\sqrt{\kappa}_i|v|_{1,2,\tilde{w}_E\cap\Omega_i}\\
\leq\ &\bar{C}\Big\{\sum_{T\in\mathcal{T}_h}\frac{h_T^2}{\kappa_T}||f+\kappa_T\Delta u_h||^2_{0,2,T}+\sum_{E\in\mathcal{E}(T)\cap\mathcal{E}_N}\frac{h_E}{\kappa_T}||g-\kappa_T\partial_n u_h||^2_{0,2,E}\\
&+\sum_{E\in\mathcal{E}(T)\cap\mathcal{E}_h}\frac{\alpha(T,E)^2h_E}{\kappa_T}||[\kappa\partial_n u_h]||^2_{0,2,E}\Big\}^{1/2}\\
&\cdot\Big\{\sum_i\Big(\sum_{T\in\mathcal{T}_h\cap\Omega_i}\kappa_i|v|^2_{1,2,\tilde{w}_T\cap\Omega_i}+\sum_{E\in((\mathcal{E}_h\backslash\mathcal{E}_{12})\cup\mathcal{E}_N)\cap\Omega_i}\kappa_i|v|^2_{1,2,\tilde{w}_E\cap\Omega_i}\Big)\\
&+\sum_{E\in E_{12}}\big(\kappa_1|v|^2_{1,2,\tilde{w}_E\cap\Omega_1}+\kappa_2|v|^2_{1,2,\tilde{w}_E\cap\Omega_2}\big)\Big\}^{1/2}\\
\leq\ &\bar{C}\Big\{\sum_{T\in\mathcal{T}_h}\eta^2_{R,T}+\frac{h_T^2}{\kappa_T}||f_T-f||^2_{0,2,T}+\sum_{E\in\mathcal{E}_N\cap\mathcal{E}(T)}\frac{h_E}{\kappa_T}||g_E-g||^2_{0,2,E}\Big\}^{1/2}||v||_{e,\Omega}\,,
\end{aligned}
$$

where

$$\bar{C} := \max\{C_6, \sqrt{2}C_7, C_\sigma\}\,.$$

Here C_σ is a positive constant which counts the number of times a given element of the triangulation of Ω_i is recovered by neighborhoods $\tilde{w}_T$ or $\tilde{w}_E$. To obtain (3.6), we just have to take $v = u - u_h$ and divide each term of the above inequality by $||u-u_h||_{e,\Omega}$, completing then the proof of step 2.

Step 3. Inverse bound.

Let us consider the local bubble test function $v_T := (f_T + \kappa_T\Delta u_h)b_T$. Then using the equivalence of norms on $\mathbb{P}_{k-2}(T)$, we obtain

$$\int_T (f_T+\kappa_T\Delta u_h)v_T \geq C_1||f_T+\kappa_T\Delta u_h||^2_{0,2,T}\,. \tag{3.10}$$

On the other hand, since the support of the function v_T is included in T, we have after integrating by parts and using the fact that u is a solution of the continuous problem

$$\begin{aligned}
\int_T (f_T + \kappa_T \Delta u_h) v_T &= \int_T f v_T + \int_T \kappa_T \Delta u_h v_T + \int_T (f_T - f) v_T \\
&= \int_\Omega f v_T + \int_{\Gamma_N} g v_T - \int_\Omega \kappa \nabla u_h \cdot \nabla v_T + \int_T (f_T - f) v_T \\
&= \int_T \kappa_T \nabla (u - u_h) \cdot \nabla v_T + \int_T (f_T - f) v_T \\
&\leq \sqrt{\kappa_T} |u - u_h|_{1,2,T} \sqrt{\kappa_T} ||\nabla v_T||_{0,2,T} + ||f_T - f||_{0,2,T} ||v_T||_{0,2,T} \,.
\end{aligned}$$

By using now the discrete inverse inequalities on $\mathbb{P}_{k-2}(T)$ and the equivalence of L^2 norms on $\mathbb{P}_{k-2}(T)$

$$\begin{aligned}
||\nabla v_T||_{0,2,T} &\leq \sqrt{C_3} h_T^{-1} ||f_T + \kappa_T \Delta u_h||_{0,2,T}, \\
||v_T||^2_{0,2,T} &\leq C_2 ||f_T + \kappa_T \Delta u_h||^2_{0,2,T},
\end{aligned}$$

we get

$$\begin{aligned}
\int_T (f_T + \kappa_T \Delta u_h) v_T &\leq ||u - u_h||_{e,T} \sqrt{C_3} h_T^{-1} \kappa_T^{1/2} ||f_T + \kappa_T \Delta u_h||_{0,2,T} \\
&\quad + \sqrt{C_2} ||f_T - f||_{0,2,T} ||f_T + \kappa_T \Delta u_h||_{0,2,T} \\
&\leq ||f_T + \kappa_T \Delta u_h||_{0,2,T} \Big(\sqrt{C_3} h_T^{-1} \kappa_T^{1/2} ||u - u_h||_{e,T} \\
&\quad + \sqrt{C_2} ||f - f_T||_{0,2,T} \Big) \,.
\end{aligned}$$

Hence, combining the two inequalities, we finally obtain

$$||f_T + \kappa_T \Delta u_h||_{0,2,T} \leq C_8 \Big\{ h_T^{-1} \kappa_T^{1/2} ||u - u_h||_{e,T} + ||f_T - f||_{0,2,T} \Big\} \,, \tag{3.11}$$

with

$$C_8 := \frac{\max\{\sqrt{C_2}, \sqrt{C_3}\}}{C_1}.$$

Next, we consider an arbitrary boundary face $E \in \mathcal{E}_N$ and define

$$v_E := (g_E - \kappa_T \partial_n u_h) b_E \,.$$

From our norm equivalence in $\mathbb{P}_{k-1}(E)$, we first have

$$\int_E (g_E - \kappa_T \partial_n u_h) v_E \geq C_1 ||g_E - \kappa_T \partial_n u_h||^2_{0,2,E}. \tag{3.12}$$

On the other hand, after integration by part, and using the construction of u, v_E and our basic inverse inequalities, we obtain

$$\begin{aligned}
&\int_E (g_E - \kappa_T \partial_n u_h) v_E \\
= \;& \int_E (g - \kappa_T \partial_n u_h) v_E + \int_E (g_E - g) v_E \\
= \;& \int_\Omega f v_E + \int_{\Gamma_N} g v_E - \int_\Omega \kappa_T \nabla u_h \cdot \nabla v_E - \sum_{T' \in w_E} \int_{T'} (f + \kappa \Delta u_h) v_E + \int_E (g_E - g) v_E \\
= \;& \int_{w_E} \kappa_T \nabla (u - u_h) \cdot \nabla v_E - \sum_{T' \in w_E} \int_{T'} (f + \kappa \Delta u_h) v_E + \int_E (g_E - g) v_E \\
\leq \;& \sqrt{\kappa_T} |u - u_h|_{1,2,w_E} \sqrt{\kappa_T} ||\nabla v_E||_{0,2,w_E} + \sum_{T' \in w_E} ||f + \kappa \Delta u_h||_{0,2,T'} ||v_E||_{0,2,T'} \\
& + ||g_E - g||_{0,2,E} ||v_E||_{0,2,E} \\
\leq \;& ||u - u_h||_{e,w_E} \sqrt{C_4} h_E^{-1/2} \kappa_T^{1/2} ||g_E - \kappa_T \partial_n u_h||_{0,2,E} \\
& + \sqrt{C_5} h_E^{1/2} \sum_{T' \in w_E} ||f + \kappa \Delta u_h||_{0,2,T'} ||g_E - \kappa_T \partial_n u_h||_{0,2,E} \\
& + \sqrt{C_5} ||g_E - g||_{0,2,E} ||g_E - \kappa_T \partial_n u_h||_{0,2,E} \\
\leq \;& ||g_E - \kappa_T \partial_n u_h||_{0,2,E} \Big(\sqrt{C_4} h_E^{-1/2} \kappa_T^{1/2} ||u - u_h||_{e,w_E} \\
& + \sqrt{C_5} h_E^{1/2} \sum_{T' \in w_E} ||f + \kappa \Delta u_h||_{0,2,T'} + \sqrt{C_5} ||g_E - g||_{0,2,E} \Big).
\end{aligned}$$

Thus, by combining the above two inequalities, we finally get

$$\begin{aligned}
||g_E - \kappa_T \partial_n u_h||_{0,2,E} \;\leq\; & C_9 \Big\{ h_E^{-1/2} \kappa_T^{1/2} ||u - u_h||_{e,w_E} \\
& + h_E^{1/2} \sum_{T' \subset w_E} (||f - f_{T'}||_{0,2,T'} + ||f_{T'} + \kappa \Delta u_h||_{0,2,T'}) \\
& + ||g_E - g||_{0,2,E} \Big\}.
\end{aligned} \tag{3.13}$$

with

$$C_9 = \frac{\max\{\sqrt{C_4}, \sqrt{C_5}\}}{C_1}$$

Finally, let us consider an internal face $E \in \mathcal{E}_h$ separating the elements T_1 and T_2 and define

$$v_E := (\frac{\alpha^2(T_1,E)}{\kappa_{T_1}} + \frac{\alpha^2(T_2,E)}{\kappa_{T_2}})^{1/2}(\kappa_{T_1}\partial_n u_h - \kappa_{T_2}\partial_n u_h)b_E\,.$$

From the equivalence of L^2 norms on $\mathbb{P}_{k-1}(E)$, we first have

$$\int_E (\kappa_{T_1}\partial_n u_h - \kappa_{T_2}\partial_n u_h)v_E \geq C_1(\frac{\alpha^2(T_1,E)}{\kappa_{T_1}} + \frac{\alpha^2(T_2,E)}{\kappa_{T_2}})^{1/2}||[\kappa\partial_n u_h]||^2_{0,2,E}\,. \tag{3.14}$$

On the other hand, by integration by parts, using the fact that the support of v_E is included in w_E, the construction of u and the equivalence of norms

$$\begin{aligned}
&\int_E (\kappa_{T_1}\partial_n u_h - \kappa_{T_2}\partial_n u_h)v_E \\
= \;& \sum_{T\subset w_E}\int_T (f+\kappa_T\Delta u_h)v_E - \int_\Omega f v_E - \int_{\Gamma_N} g v_E + \int_\Omega \kappa\nabla u_h\cdot\nabla v_E \\
= \;& \sum_{T\subset w_E}\int_T (f+\kappa_T\Delta u_h)v_E - \sum_i \int_{w_E\cap\Omega_i}\kappa_i\nabla(u-u_h)\cdot\nabla v_E \\
\leq \;& \sum_i \sqrt{\kappa_i}|u-u_h|_{1,2,w_E\cap\Omega_i}\sqrt{\kappa_i}||\nabla v_E||_{0,2,w_E\cap\Omega_i} \\
& +||(f+\kappa\Delta u_h)||_{0,2,w_E\cap\Omega_i}||v_E||_{0,2,w_E\cap\Omega_i} \\
\leq \;& \sum_i \sqrt{\kappa_i}||u-u_h||_{e,w_E\cap\Omega_i}\sqrt{C_4}h_E^{-1/2}||v_E||_{0,2,E} \\
& +||f+\kappa\Delta u_h||_{0,2,w_E\cap\Omega_i}||v_E||_{0,2,w_E\cap\Omega_i} \\
\leq \;& \sqrt{h_E}||\kappa_1\partial_n u_h - \kappa_2\partial_n u_h||_{0,2,E}\Big(\sum_i \sqrt{C_4}h_E^{-1}||u-u_h||_{e,w_E\cap\Omega_i}\sqrt{\kappa_i}(\frac{\alpha_1^2}{\kappa_1}+\frac{\alpha_2^2}{\kappa_2})^{1/2} \\
& +\sqrt{C_5}(\frac{\alpha_1^2}{\kappa_1}+\frac{\alpha_2^2}{\kappa_2})^{1/2}||f+\kappa\Delta u_h||_{0,2,w_E\cap\Omega_i}\Big) \\
\leq \;& \sqrt{h_E}||\kappa_1\partial_n u_h - \kappa_2\partial_n u_h||_{0,2,E}\Big(\sum_i \sqrt{C_4}h_E^{-1}||u-u_h||_{e,w_E\cap\Omega_i}\sqrt{\kappa_i}(\frac{\alpha_1^2}{\kappa_1}+\frac{\alpha_2^2}{\kappa_2})^{1/2}
\end{aligned}$$

$$+\sqrt{C_5}(\frac{\alpha_1^2}{\kappa_1}+\frac{\alpha_2^2}{\kappa_2})^{1/2}\sum_{T'\in w_E}||f_{T'}+\kappa\Delta u_h||_{0,2,T'}$$
$$+\sqrt{C_5}(\frac{\alpha_1^2}{\kappa_1}+\frac{\alpha_2^2}{\kappa_2})^{1/2}\sum_{T'\in w_E}||f-f_{T'}||_{0,2,T'}\Big).$$

By construction of coefficients α_i, we have that

$$\sqrt{\kappa_i}(\frac{\alpha_1^2}{\kappa_1}+\frac{\alpha_2^2}{\kappa_2})^{1/2}\leq 1$$

and hence by using the above inequalities and (3.11) we obtain

$$(\frac{\alpha^2(T_1,E)}{\kappa_{T_1}}+\frac{\alpha^2(T_2,E)}{\kappa_{T_2}})^{1/2}||[\kappa\partial_n u_h]||_{0,2,E}$$
$$\leq C_{10}\Big(h_E^{-1/2}||u-u_h||_{e,w_E}+\sum_{T'\subset w_E}h_E^{1/2}\frac{1}{\sqrt{\kappa_{T'}}}||f-f_{T'}||_{0,2,T'}\Big) \quad (3.15)$$

with

$$C_{10}:=\frac{\max\{\sqrt{C_4},C_8\sqrt{C_5},2\sqrt{C_5}\}}{C_1}.$$

Then from (3.11), (3.13) and (3.15), we get that there exists a positive constant $\underline{C}$ such that

$$\eta_{R,T}\leq\underline{C}\Big\{||u-u_h||_{e,w_T}^2+\sum_{T'\subset w_T}\frac{h_{T'}^2}{\kappa_{T'}}||f_T-f||_{0,2,T'}^2+\sum_{E\in\mathcal{E}(T)\cap\mathcal{E}_N}\frac{h_E}{\kappa_T}||g_E-g||_{0,2,E}^2\Big\}^{1/2} \blacksquare$$

4 The elasticity problem

We now try to extend the previous approach to linear elasticity problems. Let Ω an Lipschitz, bounded domain of $\mathbb{R}^n$ with an interior boundary denoted Γ_{12}. This boundary represents the interface between two elastic, isotropic and homogeneous materials, noted Ω_1 and Ω_2, respectively (see Fig 3). Let $\Gamma := \partial\Omega$ such that $\Gamma = \Gamma_D \cup \Gamma_N$, $\Gamma_D \cap \Gamma_N = \emptyset$, with $\partial\Omega_i \cap \Gamma_D \neq \emptyset$, $i = 1, 2$. This means for the time being that we assume that each subdomain is fixed on part of its boundary. This assumption will be useful to relate the H^1 semi-norm used in the local interpolation (2.7) and the local energy norm. It can be relaxed if we can prove this interpolation result directly with the $|\boldsymbol{\epsilon}(\mathbf{v})|_{0,2}$ norm. In this framework, we consider the following elasticity problem

$$
\text{(P)} \qquad \left\{ \begin{aligned} -\operatorname{div} \boldsymbol{\sigma} &= \mathbf{f} && \text{in } \Omega\,, \\ \mathbf{u} &= 0 && \text{on } \Gamma_D\,, \\ \boldsymbol{\sigma}\cdot\mathbf{n} &= \mathbf{g} && \text{on } \Gamma_N\,, \end{aligned} \right.
$$

where $\mathbf{f} \in \mathrm{L}^2(\Omega)^n$ and $\mathbf{g} \in \mathrm{L}^2(\Gamma_D)^n$ are the external forces and $\boldsymbol{\sigma}$ is the stress tensor. Assuming isotropy, this tensor satisfies the constitutive law

$$
\boldsymbol{\sigma} = (\sigma_{ij}) := \left\{ \begin{array}{ll} \lambda_1 \epsilon_{pp}(\mathbf{u})\delta_{ij} + 2\mu_1\epsilon_{ij}(\mathbf{u}) & \text{on } \Omega_1\,, \\ \lambda_2 \epsilon_{pp}(\mathbf{u})\delta_{ij} + 2\mu_2\epsilon_{ij}(\mathbf{u}) & \text{on } \Omega_2\,, \end{array} \right.
$$

with $\lambda_i, \mu_i > 0$ the Lame's coefficients of the material Ω_i and $\epsilon_{ij}(\mathbf{u}) := \frac{1}{2}(u_{i,j} + u_{j,i})$ the components of the linearized strain tensor $\boldsymbol{\epsilon}(\mathbf{u})$ associated to $\mathbf{u}$.

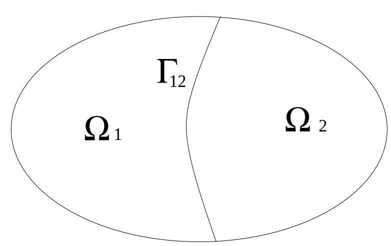


Figure 3: $\Omega = \Omega_1 \cup \Gamma_{12} \cup \Omega_2$.

The standard weak formulation of the problem (P) is then:
Find $\mathbf{u} \in \mathbf{H}$ *such that*

$$
a(\mathbf{u}, \mathbf{v}) = < F, \mathbf{v} > \quad , \forall \mathbf{v} \in \mathbf{H} \tag{4.1}
$$

where

$$\begin{aligned} \mathbf{H} &:= \{\mathbf{v} \in \mathrm{H}^1(\Omega)^n \,/\, \mathbf{v} = \mathbf{0} \text{ on } \Gamma_D\} \\ a(\mathbf{u},\mathbf{v}) &:= \int_{\Omega_1} \boldsymbol{\sigma}_1(\mathbf{u}) : \boldsymbol{\epsilon}(\mathbf{v}) + \int_{\Omega_2} \boldsymbol{\sigma}_2(\mathbf{u}) : \boldsymbol{\epsilon}(\mathbf{v}) \\ < F, \mathbf{v} > &:= \int_{\Omega} \mathbf{f} \cdot \mathbf{v} + \int_{\Gamma_N} \mathbf{g} \cdot \mathbf{v}\,. \end{aligned}$$

Let $\mathbf{H}_h$ the finite element space defined by

$$\mathbf{H}_h := \{\mathbf{v}_h \in \mathcal{C}(\Omega)^n \,/\, \mathbf{v}_h = \mathbf{0} \text{ on } \Gamma_D\,, \forall T \in \mathcal{T}_h\,, \mathbf{v}_h|_T \in \mathbb{P}_k(T)^n\,\} \quad (k \geq 1)\,.$$

Then the approximate problem of (4.1) is again:
Find $\mathbf{u}_h \in \mathbf{H}_h$ *such that*

$$a(\mathbf{u}_h, \mathbf{v}_h) = < F, \mathbf{v}_h > \quad , \forall \mathbf{v}_h \in \mathbf{H}_h\,. \tag{4.2}$$

We define the energy norm by:

$$\begin{aligned} ||\mathbf{v}||_{e,\Omega} &:= \{||\mathbf{v}||^2_{e,\Omega_1} + ||\mathbf{v}||^2_{e,\Omega_2}\}^{1/2} \\ &= \{\int_{\Omega_1} \boldsymbol{\sigma}_1(\mathbf{v}) : \boldsymbol{\epsilon}(\mathbf{v}) + \int_{\Omega_2} \boldsymbol{\sigma}_2(\mathbf{v}) : \boldsymbol{\epsilon}(\mathbf{v})\}^{1/2} \quad , \forall \mathbf{v} \in \mathbf{H}\,. \end{aligned} \tag{4.3}$$

By the Korn's inequality, since $\partial\Omega_i \cap \Gamma_D$ has a non empty measure, there exist two positive constants C_{Ω_1} and C_{Ω_2}, depending only on the geometry of Ω_1 and Ω_2, respectively, such that

$$||\mathbf{v}||_{1,2,\Omega_i} \leq C_{\Omega_i} ||\boldsymbol{\epsilon}(\mathbf{v})||_{0,2,\Omega_i} \quad \forall \mathbf{v} \in \mathbf{H},\ i = 1, 2\,. \tag{4.4}$$

Finally, there exits an interpolation operator (see Lemma 2) $R_h : \mathbf{H} \to \mathbf{H}_h$ such that for all $\mathbf{v} \in \mathbf{H}$

$$\begin{aligned} ||\mathbf{v} - R_h\mathbf{v}||_{0,2,T} &\leq C_7 h_T |\mathbf{v}|_{1,2,\tilde{w}_T \cap \Omega_i} \\ ||\mathbf{v} - R_h\mathbf{v}||_{0,2,E} &\leq C_8 h_E^{1/2} |\mathbf{v}|_{1,2,\tilde{w}_E \cap \Omega_i} \end{aligned} \tag{4.5}$$

where $T \in \mathcal{T}_h$, $E \in \mathcal{E}_N \cup \mathcal{E}_h$ and the positive constants C_7, C_8 are independent of h.

With the above definitions we prove exactly the same results as in section 3. We only have to pay attention to:

- replace $\sqrt{\kappa_T}$ by $\sqrt{E_T}$ where E_T is the Young modulus of the material that composes the element T,

- use the property $0 < \nu < 0.5$ for to ensure that $1 < 1 + \nu < 3/2$,

- In the proof of step 3 we find a factor $\frac{1}{1-2\nu_i}$ that we include in a constant. It's clear that if a material is quasi-incompressible then this constant explodes. This means that our developpement is valid only for compressible materials. In fact, for practical purpose we assume that $0 < \nu \leq 0.45$.

Altogether, defining the *local residual* by

$$\begin{aligned}
\eta_{R,T} \ := \ & \Big\{ \frac{h_T^2}{E_T} ||\mathbf{f}_T + \nabla \cdot \boldsymbol{\sigma}(\mathbf{u}_h)||_{0,2,T}^2 + \sum_{E \in \mathcal{E}(T) \cap \mathcal{E}_N} \frac{h_E}{E_T} ||\mathbf{g}_E - \boldsymbol{\sigma}(\mathbf{u}_h) \cdot \mathbf{n}||_{0,2,E}^2 \\
+ \ & \sum_{E \in \mathcal{E}(T) \cap \mathcal{E}_h} \frac{\alpha(T,E)^2}{E_T} h_E ||[\boldsymbol{\sigma}(\mathbf{u}_h) \cdot \mathbf{n}]||_{0,2,E}^2 \Big\}^{1/2}, \qquad (4.6)
\end{aligned}$$

we can prove

Theorem 5 *There exist two positive constants $\bar{C}$ and $\underline{C}$, independent of h and E_i, $(i = 1, 2)$ but depending on the compressibility factor $\frac{1}{1-2\nu_i}$, such that*

$$\begin{aligned}
||\mathbf{u} - \mathbf{u}_h||_{e,\Omega} \ \leq \ & \bar{C} \Big\{ \sum_{T \in \mathcal{T}_h} \eta_{R,T}^2 + \sum_{T \in \mathcal{T}_h} \frac{h_T^2}{E_T} ||\mathbf{f}_T - \mathbf{f}||_{0,2,T}^2 \\
+ \ & \sum_{E \in \mathcal{E}_N} \frac{h_E}{E_T} ||\mathbf{g}_E - \mathbf{g}||_{0,2,E}^2 \Big\}^{1/2}, \qquad (4.7)
\end{aligned}$$

$$\eta_{R,T} \leq \underline{C} \Big\{ ||\mathbf{u} - \mathbf{u}_h||_{e,w_T}^2 + \sum_{T' \subset w_T} \frac{h_{T'}^2}{E_{T'}} ||\mathbf{f}_T - \mathbf{f}||_{0,2,T'}^2 + \sum_{E \in \mathcal{E}(T) \cap \mathcal{E}_N} \frac{h_E}{E_T} ||\mathbf{g}_E - \mathbf{g}||_{0,2,E}^2 \Big\}^{1/2}. \qquad (4.8)$$

Remark There is an extensive work relating *a posteriori* estimates and linear elasticity, for example [3], [12], [13], [14], [15], [16], [18], [21].

5 Numerical results

In this section we apply the results of section 4 to the elasticity problem

$$\begin{aligned} -\mathrm{div}\,\boldsymbol{\sigma} &= \mathbf{0} \quad \text{in } \Omega\,, \\ \mathbf{u} &= \mathbf{0} \quad \text{on } \Gamma_1\,, \\ \boldsymbol{\sigma}(\mathbf{u})\cdot\mathbf{n} &= \mathbf{P} \quad \text{on } \Gamma_2\,, \\ u_x &= 0 \quad \text{on } \Gamma_3\,, \\ \boldsymbol{\sigma}(\mathbf{u})\cdot\mathbf{n} &= \mathbf{0} \quad \text{on } \Gamma_4\,, \end{aligned}$$

with $\mathbf{P} := (0,-0.1)^T$ and $\mathbf{u} := (u_x, u_y)^T$. We try here to monitor the evolution of the residual as the mesh is refined.

Remark We follow [13] in order to obtain a optimal mesh refinement procedure, i.e. if ε_0 is the accuracy required by the user, we say that the mesh $\mathcal{T}^*$ is *optimal* if its elements number N^* is minimum and it provides a global error ε^* equal to ε_0.

In this framework, for each element $T \in \mathcal{T}$, we compute a refinement factor:

$$r_T = \frac{h_T^*}{h_T}$$

where h_T is the size of the element T of $\mathcal{T}$, and h_T^* the size of the elements of $\mathcal{T}^*$ within the T area (in 2D case).

If no strong gradients appear in the solution, then a priori error estimates indicate that the local contribution to the error should scale like

$$\frac{\eta_T^*}{\eta_T} = \left(\frac{h_T^*}{h_T}\right)^p = r_T^p$$

where p depends on the element type ($p = 1$ for linear elements, $p = 2$ for quadratics elements).

Thus we have the following minimization problem:

$$\min N^* = \sum_T \left(\frac{1}{r_T}\right)^{dim} \qquad \text{with} \qquad \sum_T r_T^2\eta_T^2 = \varepsilon_0^2\,.$$

In the 2D case, this problem admits the explicit solution:

$$r_T = \frac{\varepsilon_0^{1/p}}{\eta_T^{1/(p+1)} \left[\sum_T \eta_T^{2/(p+1)} \right]^{1/2p}} .$$

The new mesh is then obtained by a metric controlled Delaunay mesh generator [9] constrained to generate local equilateral triangles of size $r_T h_T$.

Through the section Ω_1 and Ω_4 represent two isotropic materials with $E_1 = 5$, $E_2 = 15000$ (Young's moduli), $\nu_1 = 0.45$, $\nu_2 = 0.3$ (Poisson's coefficients), and Ω_2 , Ω_3 are two orthotropic materials with elastic coefficients given by

$$\begin{pmatrix} 2000 & 50 & 0 \\ 50 & 51 & 0 \\ 0 & 0 & 2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 51 & 50 & 0 \\ 50 & 2000 & 0 \\ 0 & 0 & 2 \end{pmatrix},$$

respectively. In the first example (Fig 1), we only consider a unique isotropic material on the whole domain.

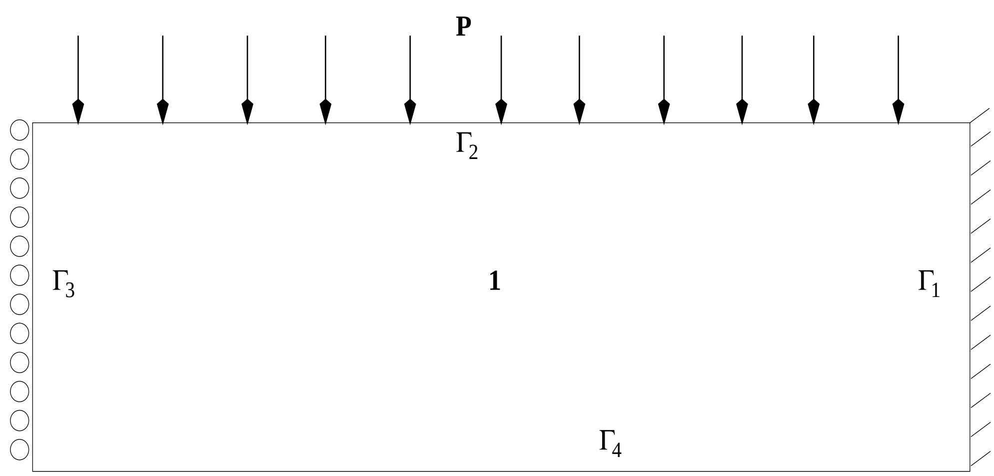
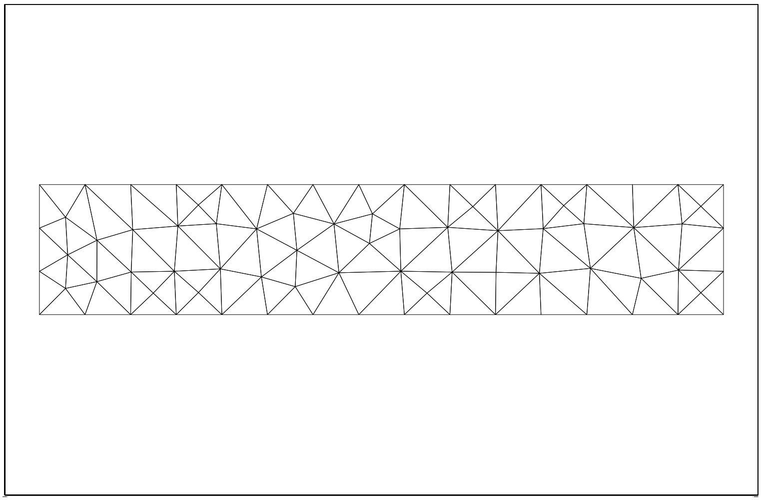


Figure 1: Example 1.

This problem is discretized using P_1 elements (3 nodes triangles), but the same type of results is valid for quadrilaterals (we have tested the same examples using Q_2 quadrilaterals). In the Figs 2-3 we can see the initial and adapted meshes and the distribution of the estimator η in these meshes.



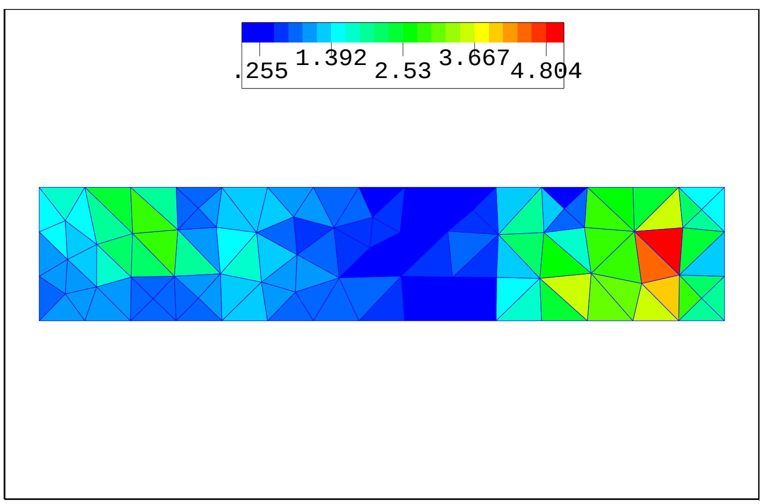


Figure 2: Initial mesh (118 elements) and distribution of the error estimator η in the Example 1.

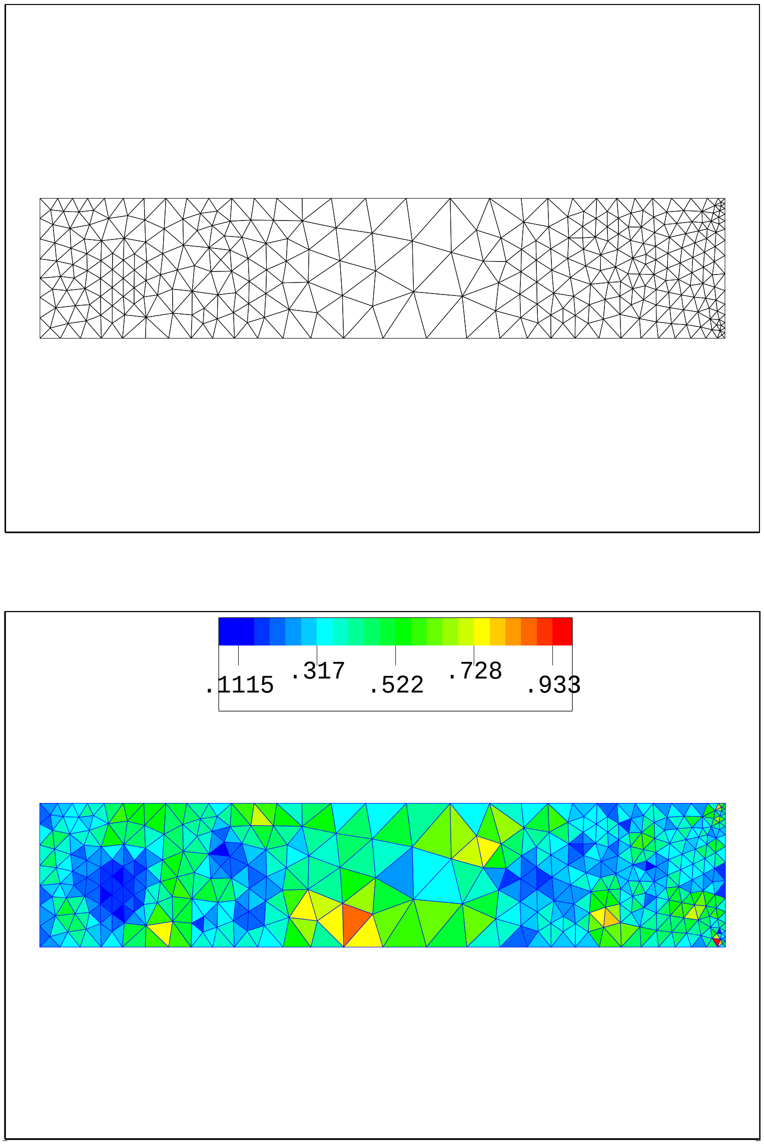


Figure 3: Final adapted mesh (571 elements) and distribution of the error estimator η in the Example 1.

In Fig 4 we present the isovalues of the solution of Example 1 and in Fig 5 we shown the Von Mises stress field for the same example.

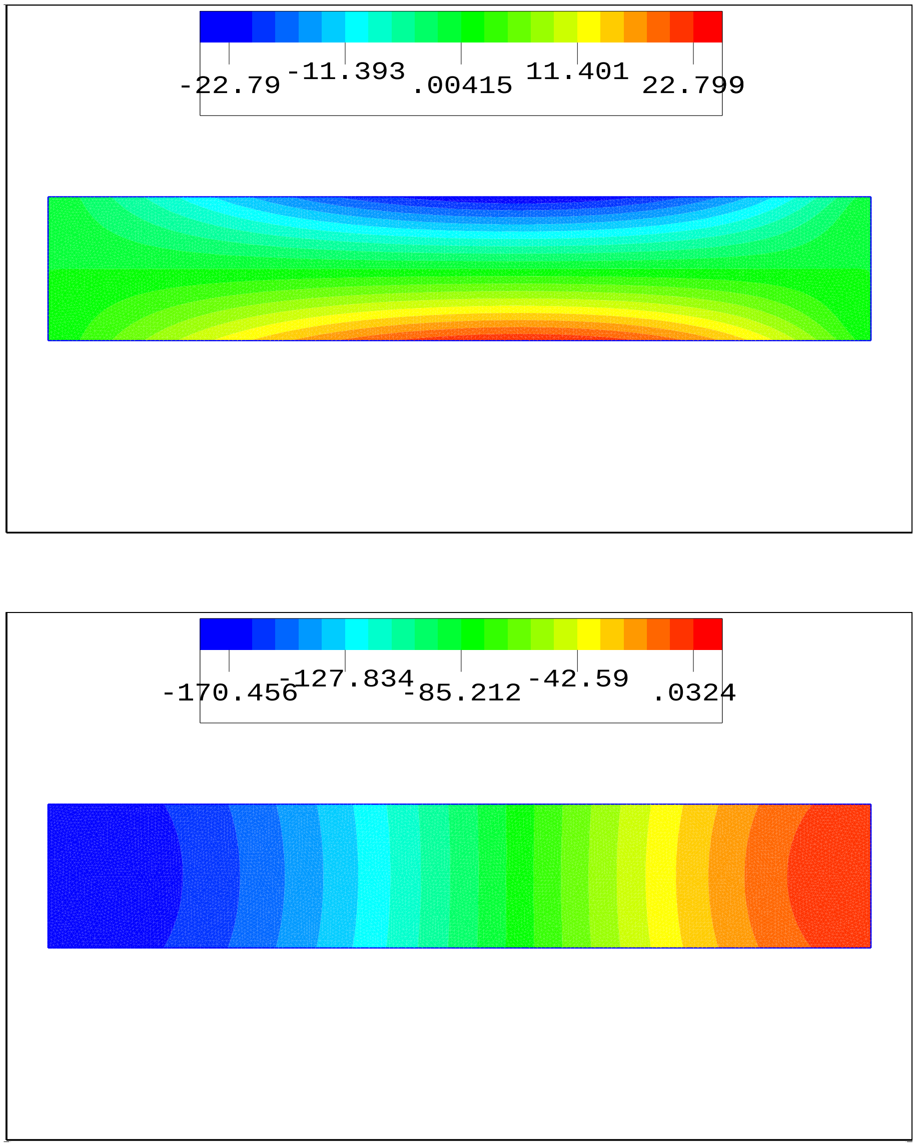


Figure 4: Isovalues of the displacement (tangential and normal) for the Example 1.

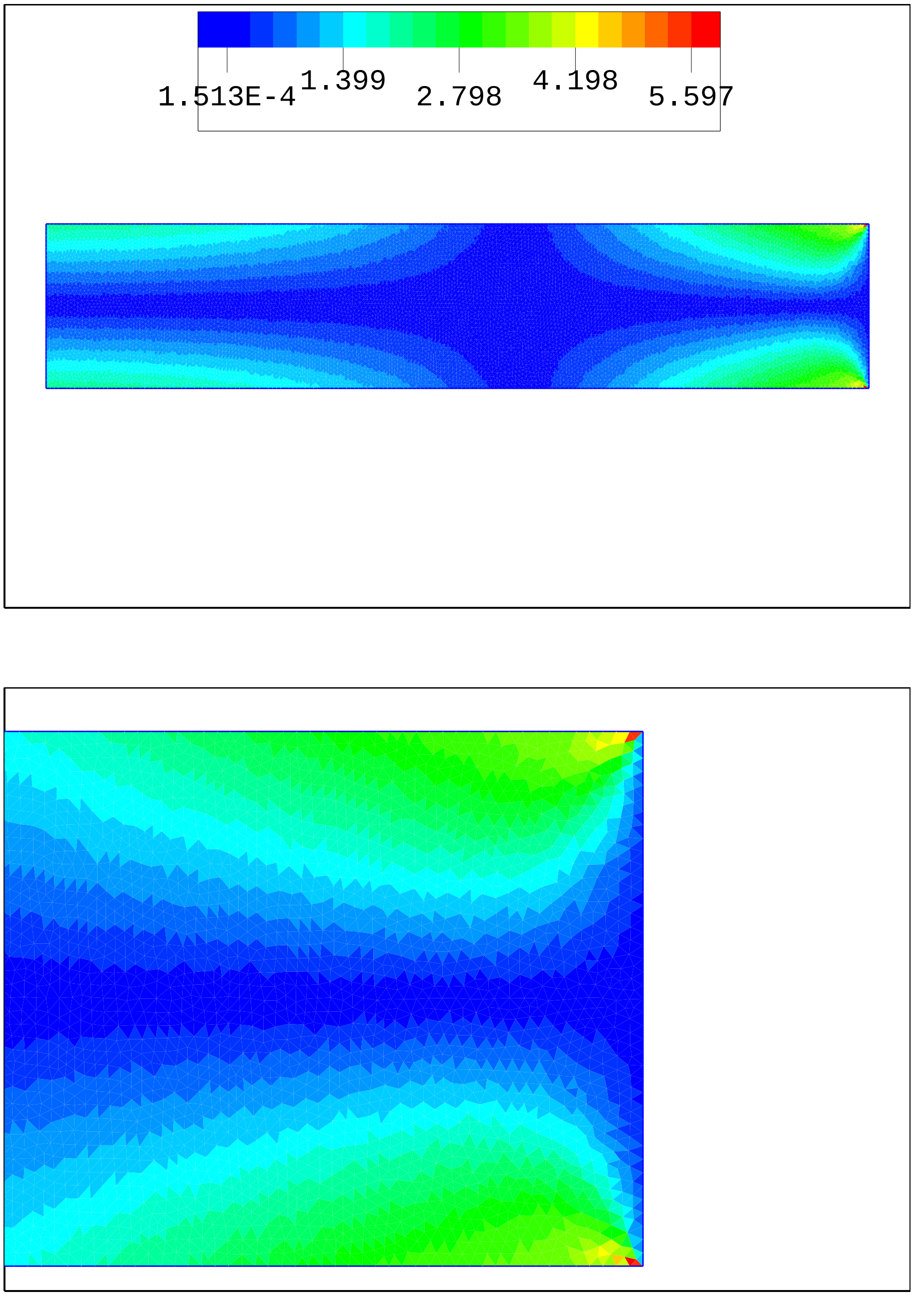


Figure 5: Global and partial view of Von Mises stress field for the Example 1.

Finally we show a comparison between the approximate solution in the initial mesh, the adapted mesh and our reference solution (calculated in a uniformly refined mesh).

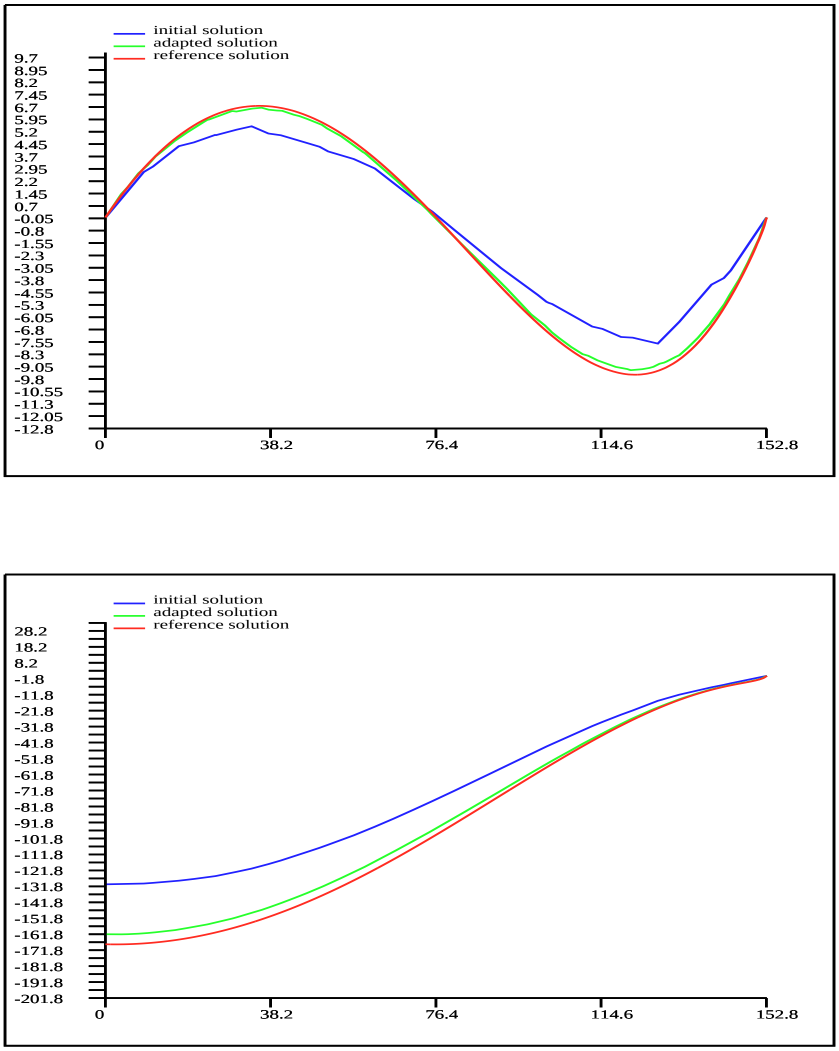


Figure 6: Comparison between the differents solutions of the Example 1 in a diagonal cut.

The last figure describes the evolution of the error estimate η.

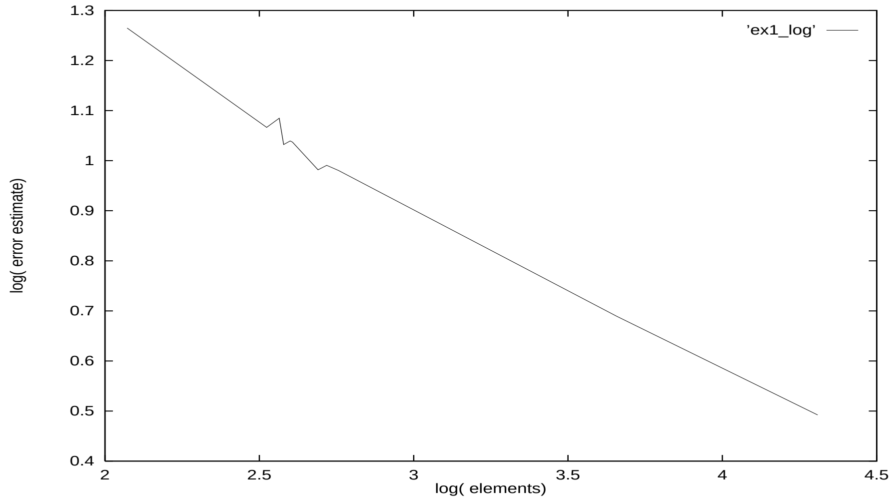


Figure 7: Residual error for the example 1.

Example 2 (Fig 8) considers a soft material 1 neighboring a more rigid isotropic material 4, discretized with the same finite element than in Example 1.

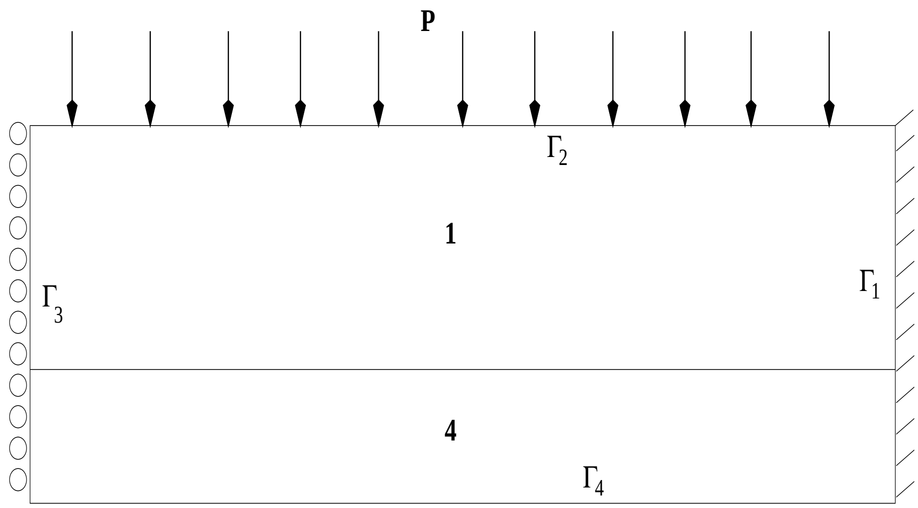


Figure 8: Example 2.

The explanation of the figures in examples 2 and 3 is the same as in Example 1.

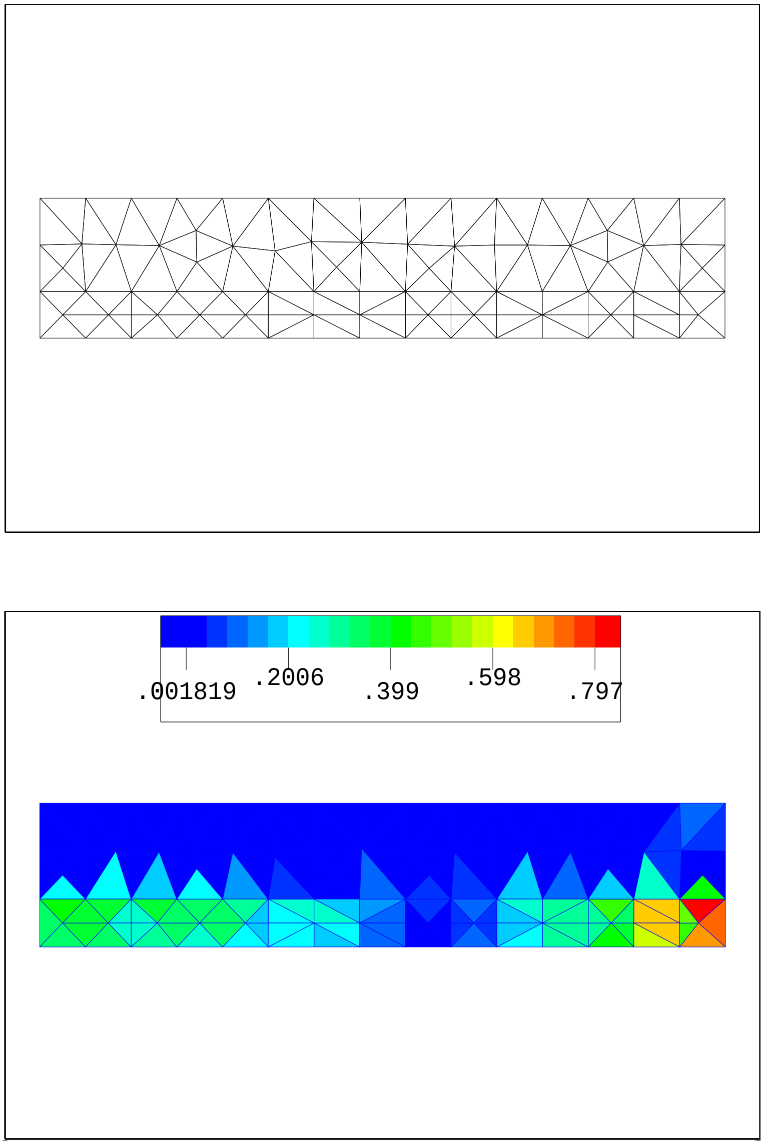


Figure 9: Initial mesh (146 elements) and distribution of the error estimator η in the Example 2.

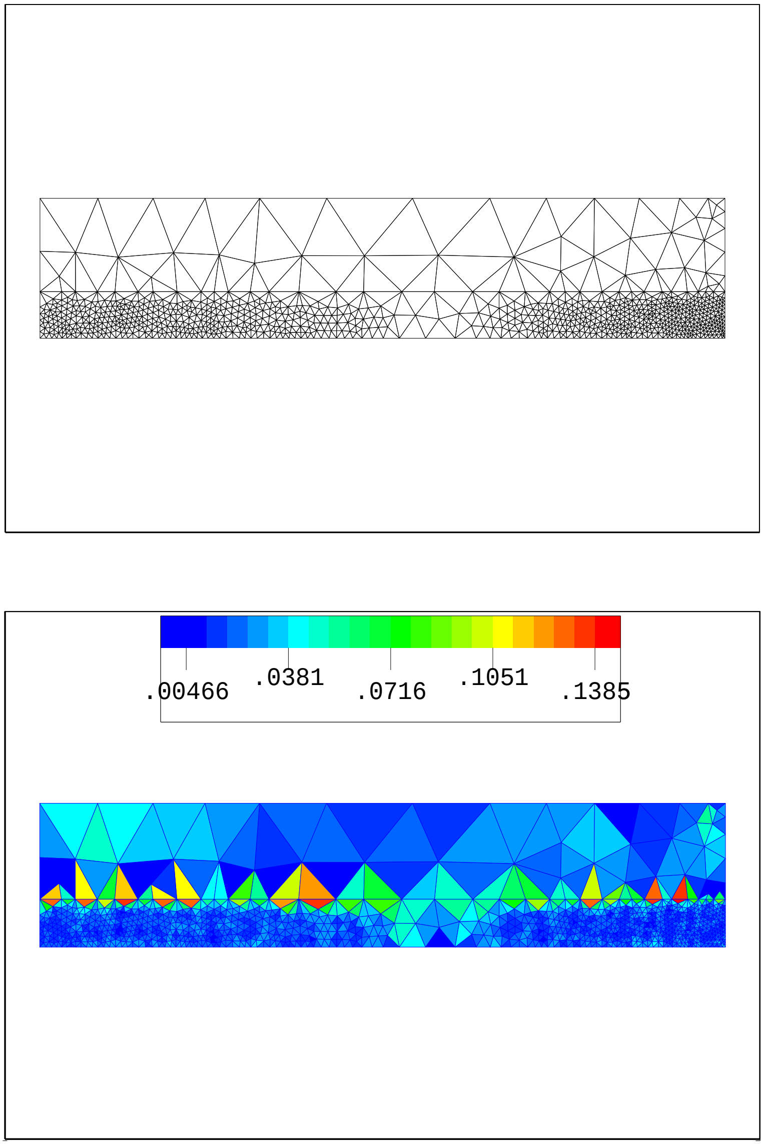


Figure 10: Final adapted mesh (1804 elements) and distribution of the error estimator η in the Example 2.

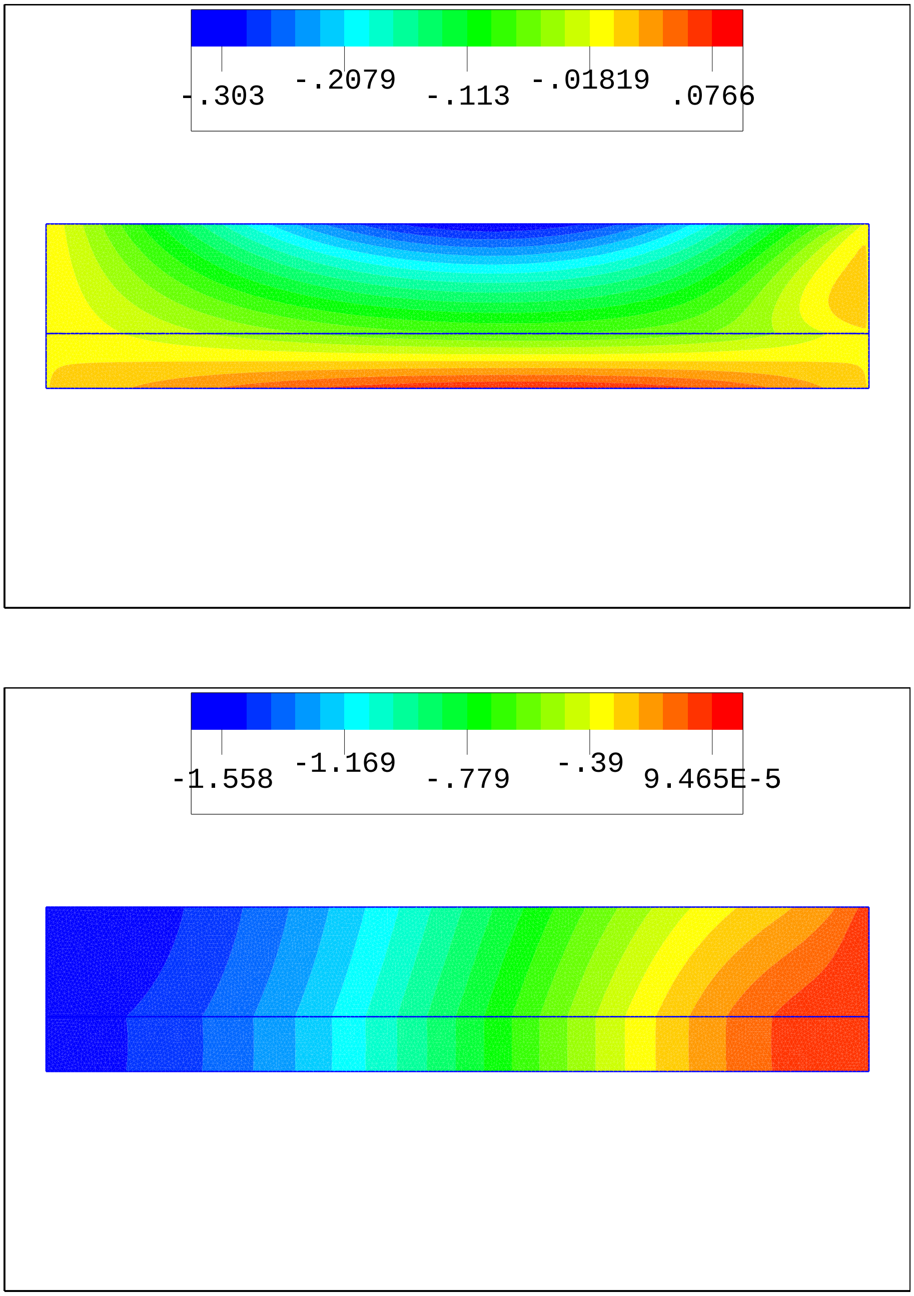


Figure 11: Isovalues of the displacement (tangential and normal) for the Example 2.

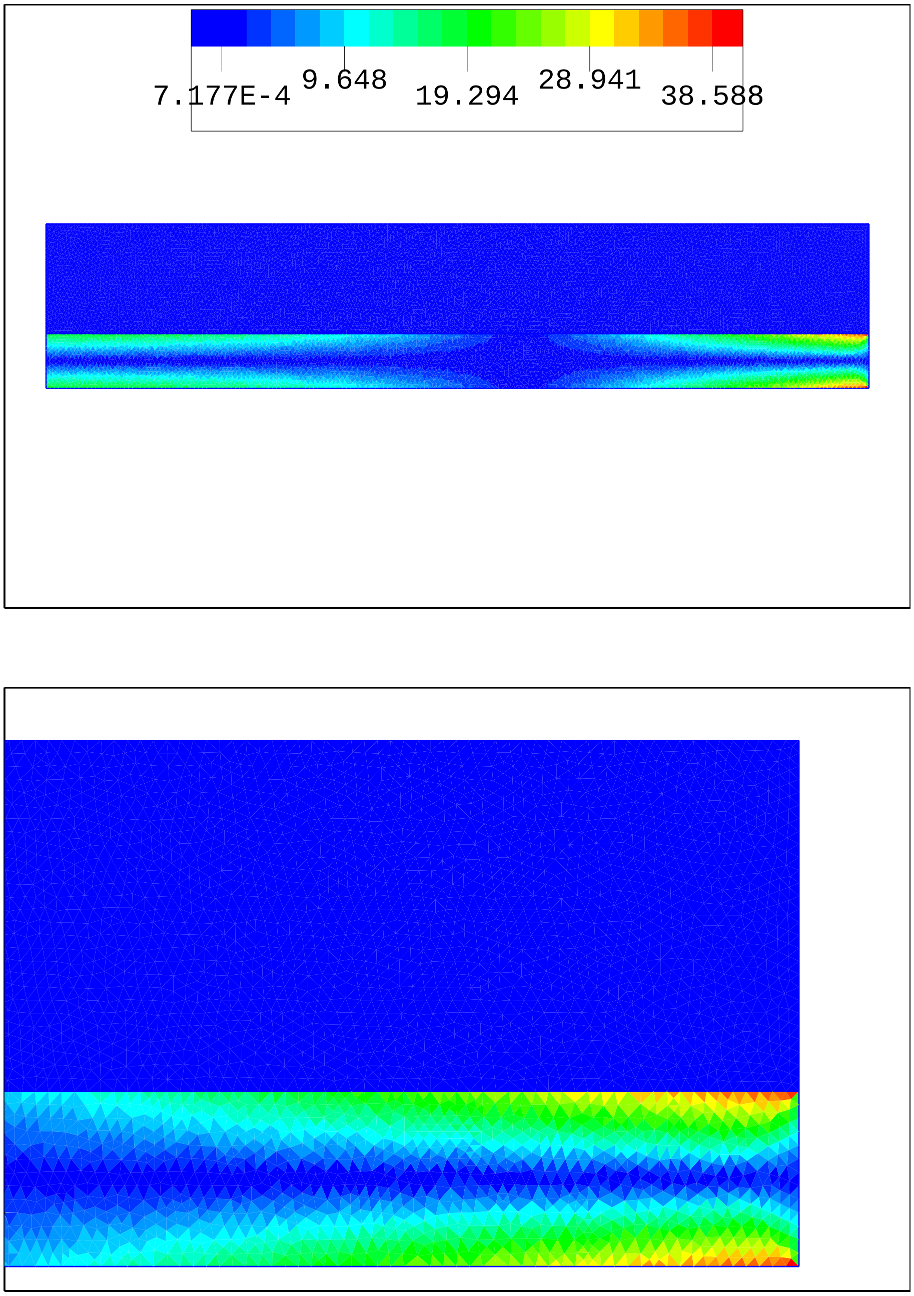
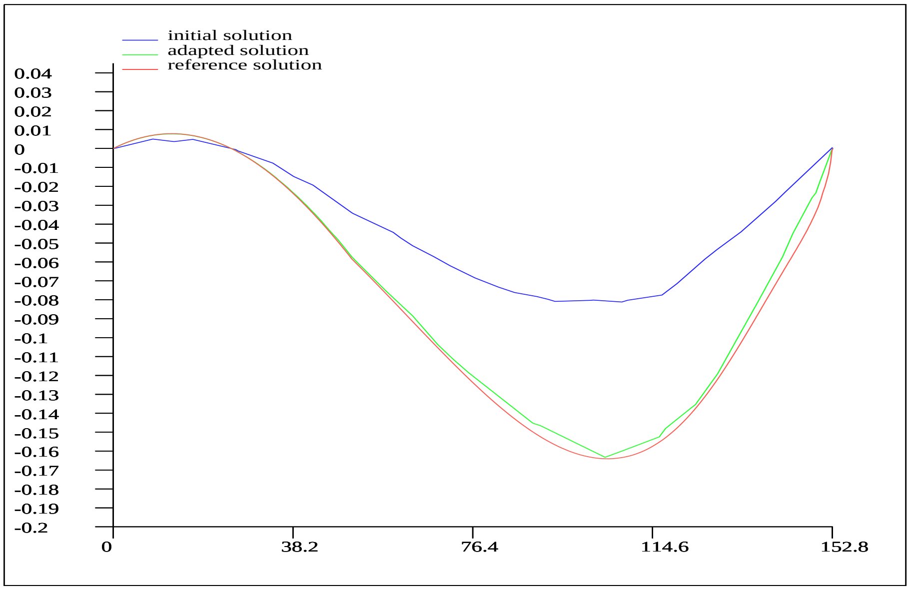


Figure 12: Global and partial view of Von Mises stress field for the Example 2.



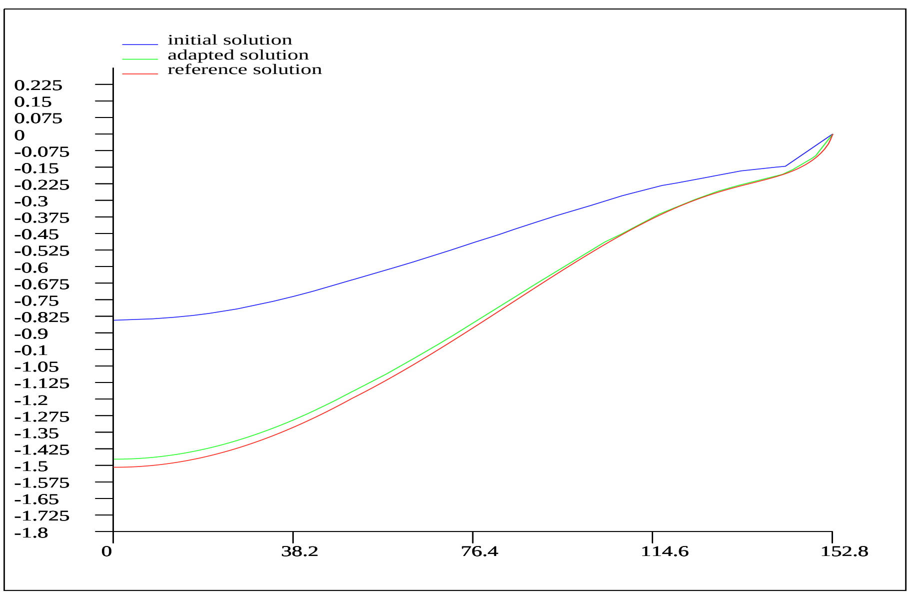


Figure 13: Comparison between the differents solutions of the Example 2 in a diagonal cut.

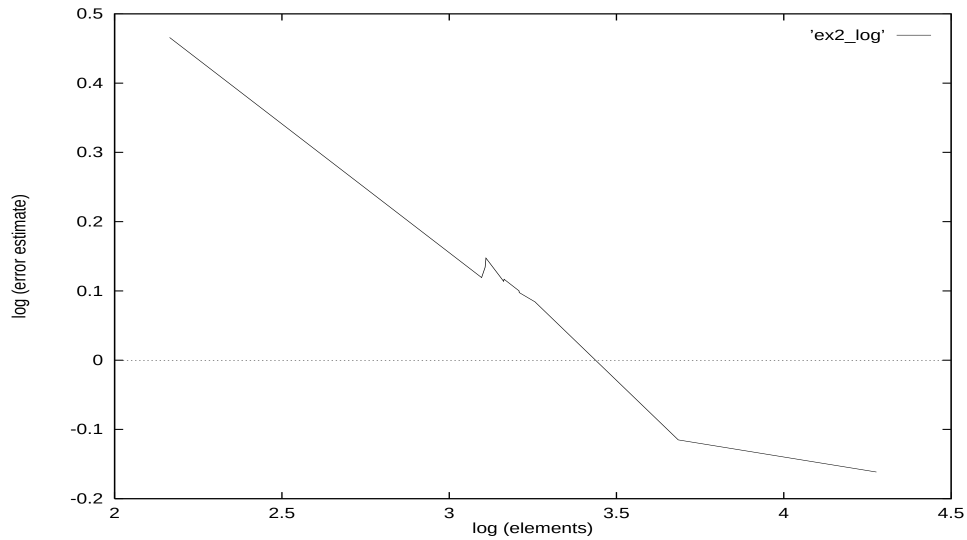


Figure 14: Residual error for the example 2.

The data for Example 3 are given on Fig 15. This represents a more realistic beam section reinforced by a rod and an anisotropic layer.

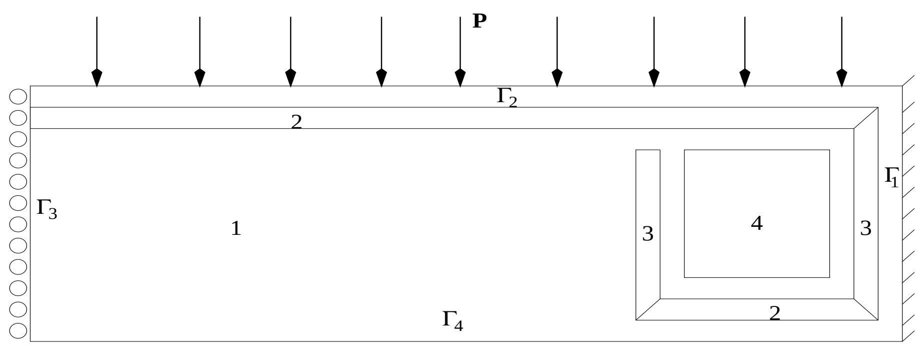


Figure 15: Example 3.

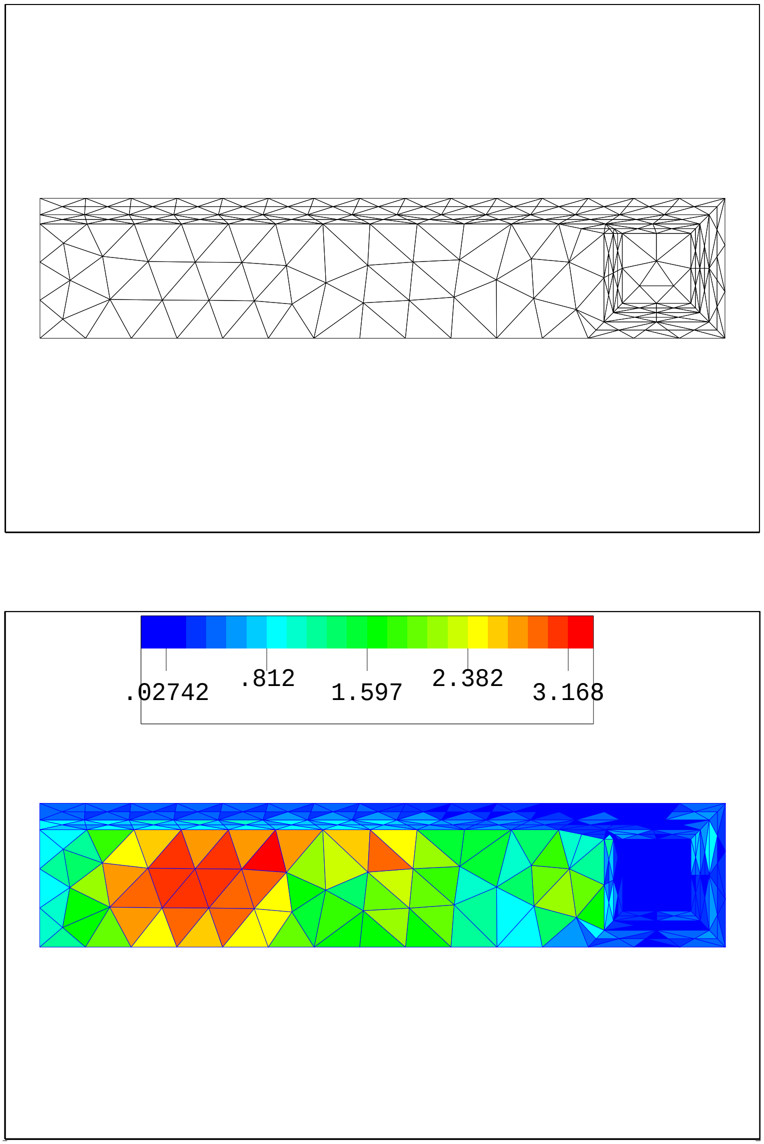


Figure 16: Initial mesh (384 elements) and distribution of the error estimator η in the Example 3.

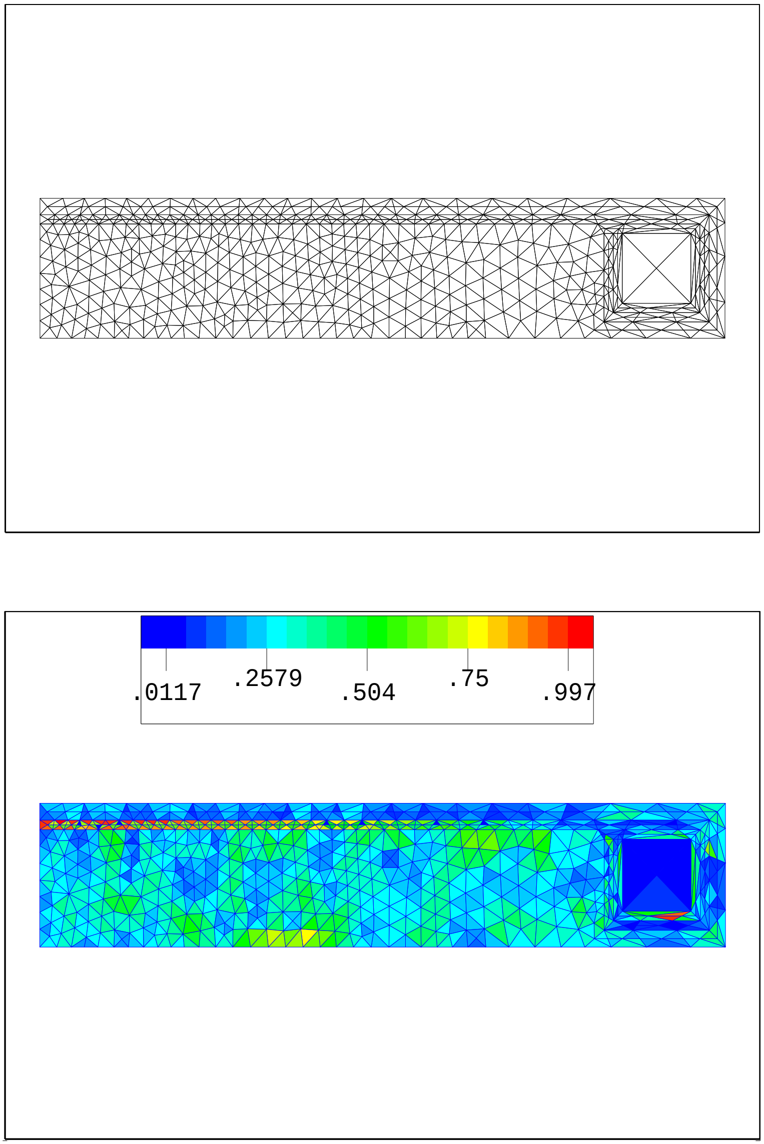


Figure 17: Final adapted mesh (1143 elements) and distribution of the error estimator η in the Example 3.

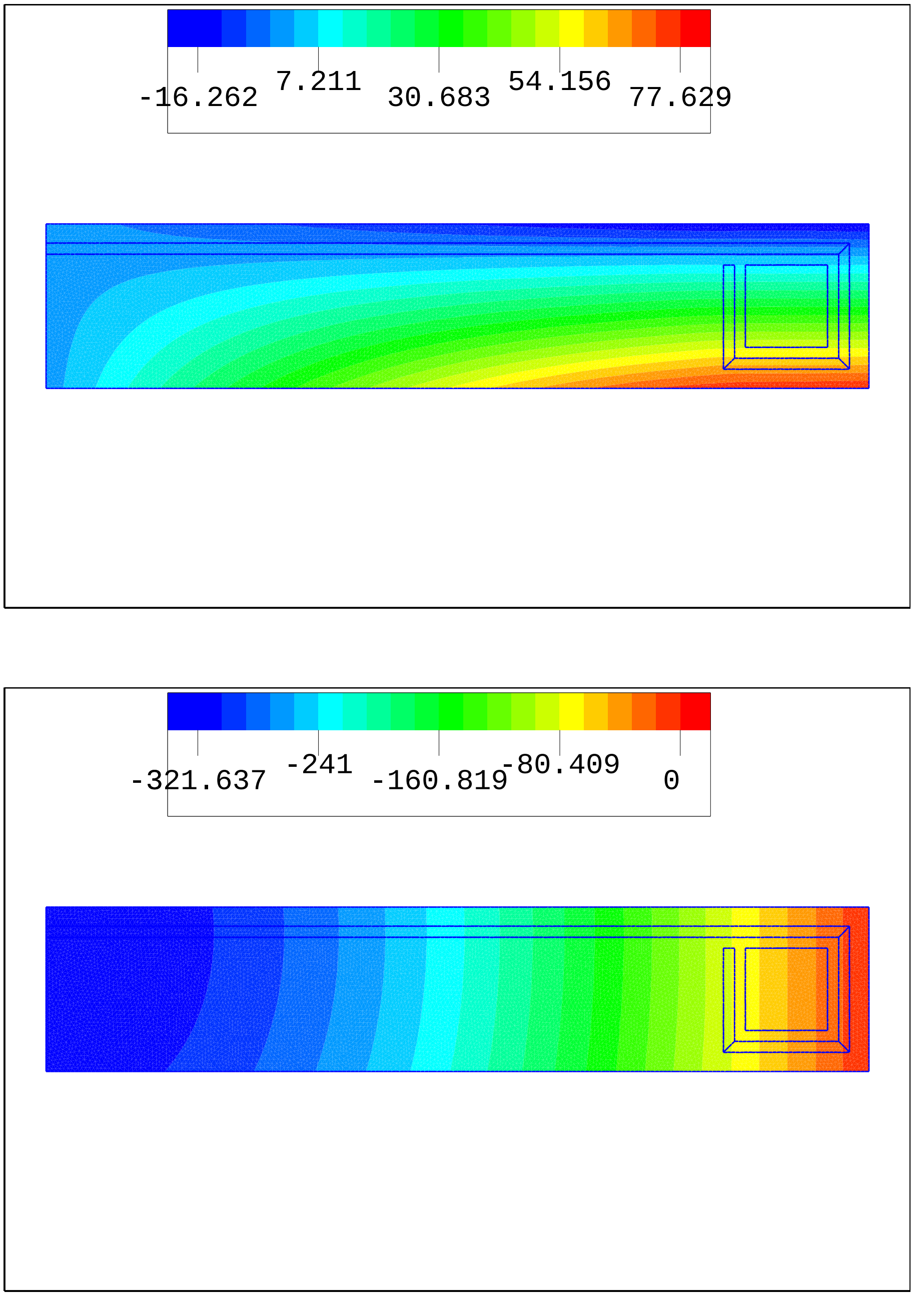


Figure 18: Isovalues of the displacement (tangential and normal) for the Example 3.

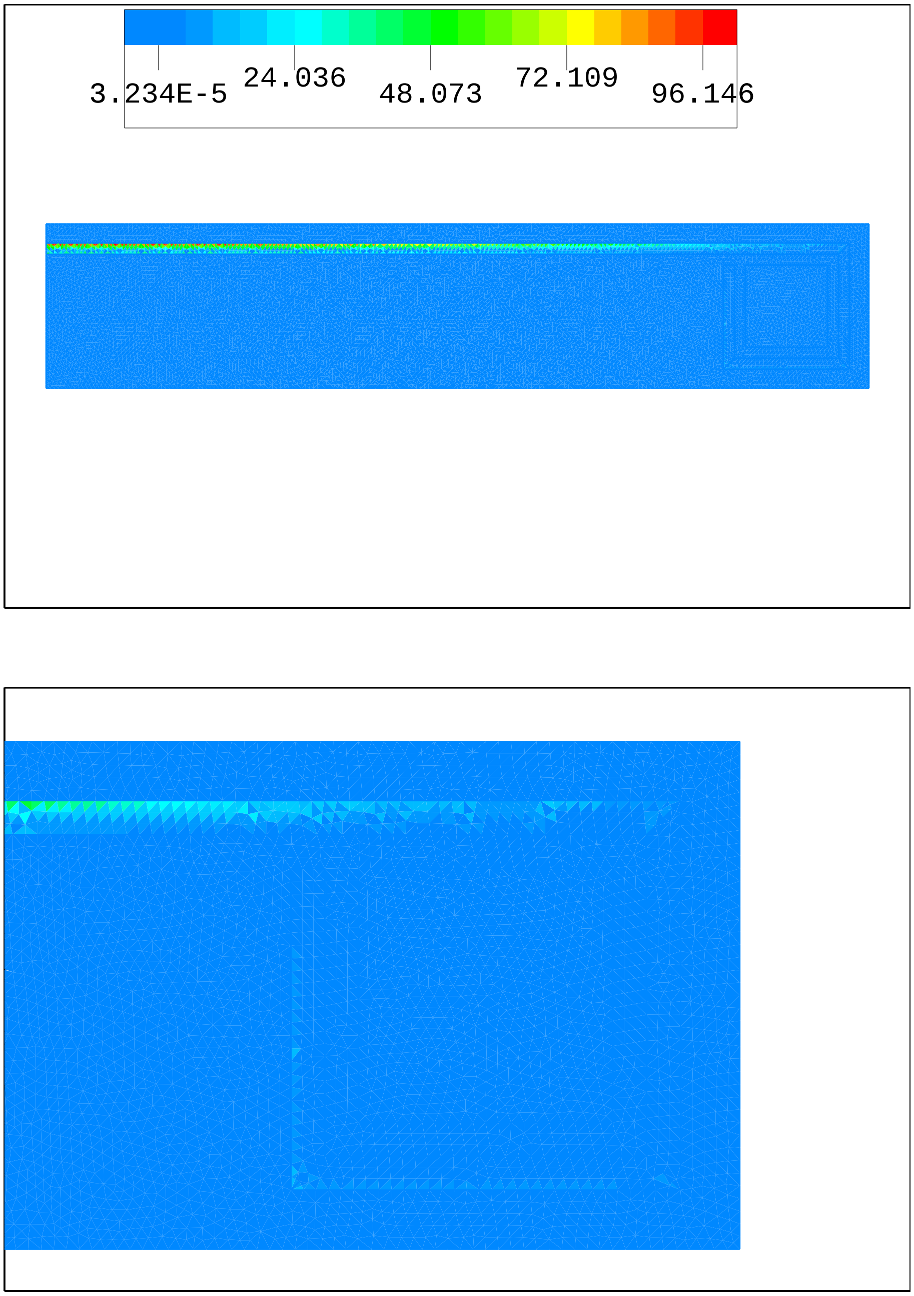
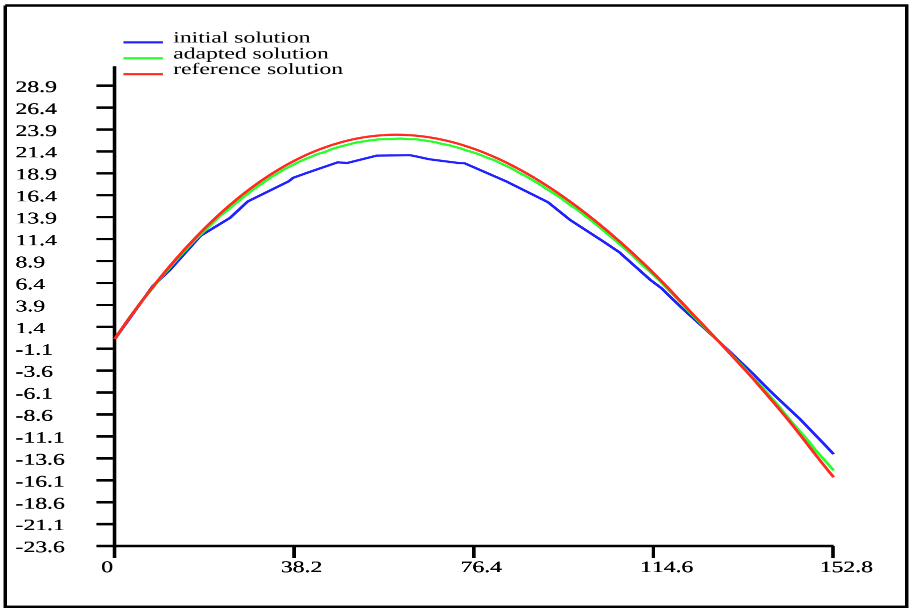


Figure 19: Global and partial view of Von Mises stress field for the Example 3.



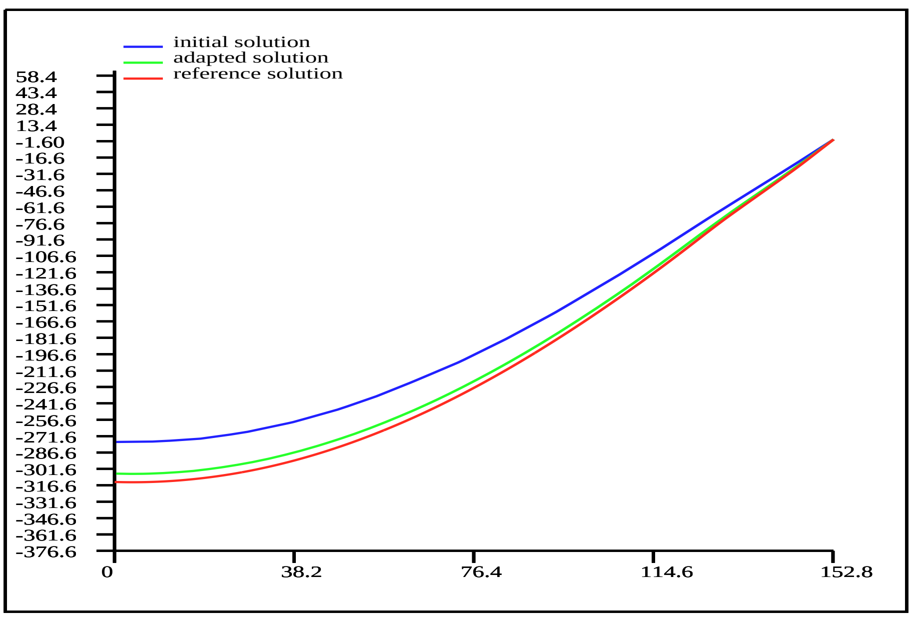


Figure 20: Comparison between the differents solutions of the Example 3 in a first diagonal cut.

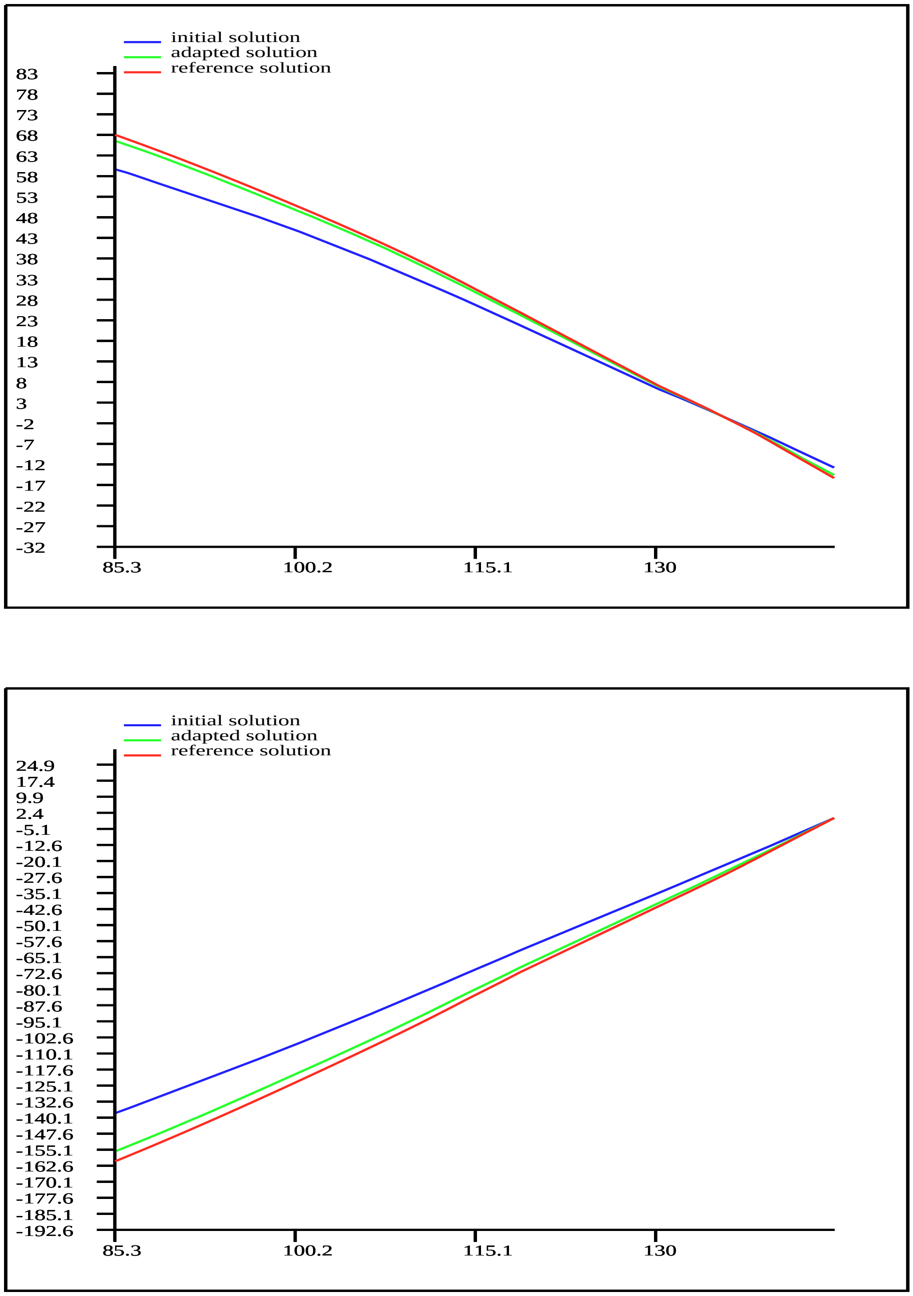


Figure 21: Comparison between the differents solutions of the Example 3 in a second diagonal cut.

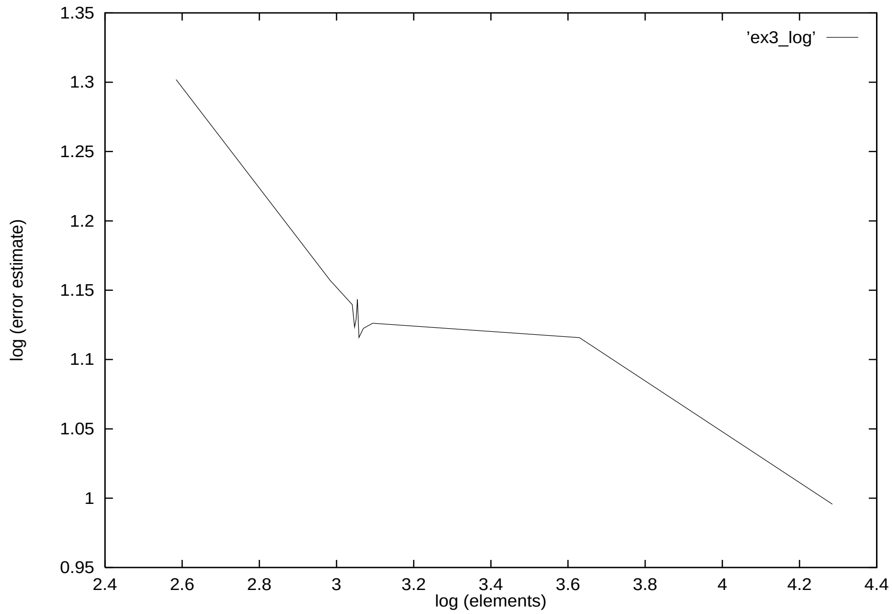


Figure 22: Residual error for the example 3.

The larger residuals appears here on the anisotropic material and next to the singularities. Recall that the proposed residual is not adapted to anisotropic situations.

6 Conclusions

We have derived and analyzed a local *a posteriori* error estimate for heterogeneous elastic bodies of residual type. The first numerical tests are encouraging for compressible isotropic materials.

Further work is needed to handle anisotropic heterogeneous bodies because we cannot proof the same type of results that for the isotropic case; nevertheless the numerical tests give the idea that our error estimate might work even in this framework. If this is not the case, it seems that one would have then to use some kind of generalization of the equilibration residual technique.

Indeed, the local H^1 norm appearing in the inverse inequality for estimating ∇v_T will no longer be uniformly equivalent to the local energy norm. We feel then that the local energy norm of the residual can only be properly obtained by solving a local Neumann problem as advocated in [2].

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